Induced Monochromatic Subconfigurations

Jaroslav Nešetřil
Department of Applied Mathematics, Charles University
Malostranské nám. 25, 118 00 Praha 1, Czech Republic
nesetril@kam.ms.mff.cuni.cz

József Solymosi
Mathematical Institute, Hungarian Academy of Sciences
H-1364 Budapest, P.O.B.127, Hungary
solymosi@cs.elte.hu

Pavel Valtr
Department of Applied Mathematics, Charles University
Malostranské nám. 25, 118 00 Praha 1, Czech Republic
valtr@kam.ms.mff.cuni.cz

Abstract

We consider induced monochromatic configurations in edge-colored planar point configurations. Contrary to the positive results for partitions of points ([NV94, NV98]) we prove two strong negative results which indicate that the Erdős–Szekeres Theorem (Happy-End Theorem) is the only positive instance of the problem.

1 Introduction

We say that a set of points in the plane is *in general position* if no three points lie on a line. A finite set of points in general position in the plane is called a *configuration*. The *orientation* of a triple (a, b, c) of non-collinear points in the plane is *positive* if the counterclockwise angle from $b - a$ to $c - a$ lies in the interval $(0, \pi)$, otherwise

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the orientation is said to be *negative*, see Fig. 1. Two configurations are said to

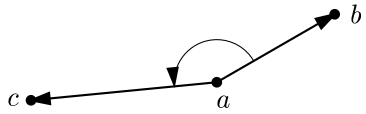

Figure 1: A positively oriented triple (a, b, c) (the triple (a, c, b) has negative orientation).

be *equivalent* if there exists a bijection between them preserving orientations of non-collinear triples of points. A configuration A *contains* a configuration B (or: B is a *subconfiguration* of A), if $B \subseteq A$. The convex hull of a configuration A is denoted by $\operatorname{conv} A$. For a point p, its x-coordinate is denoted by $x(p)$. We say that a subconfiguration B of A is *induced (in A)*, if $A \cap \operatorname{conv} B = B$. A configuration is in *convex position* if its points are vertices of a convex polygon. Induced subconfiguration of A which is in convex position is also called a *hole (in A)*. Inspired by the celebrated Erdős–Szekeres Theorem and by other results in Ramsey Theory (see [GRS80, NR90]) the following result has been (among others) shown in [NV94]:

Theorem 1 *For any configuration X and any positive integer k there exists a configuration Y such that for any coloring of ponts of Y by k colors there is a monochromatic induced subconfiguration of Y which is equivalent to X.*

In this paper we show that an analogous statement for colorings of pairs of points is false in a very strong sense. In fact we show that up to two open cases all positive instances are guaranteed by a combination of the Erdős–Szekeres and the Ramsey Theorems. More precisely this can be formulated as follows:

Let X_1, X_2, Y be configurations. We write $Y \longrightarrow (X_1, X_2)$ to denote the validity of the following statement: For any coloring of pairs of points of Y by two colors (say red and blue) Y contains an induced subconfiguration which either is equivalent to X_1 and with all pairs of its points colored by red, or is equivalent to X_2 and with all pairs of its points colored by blue. The non-validity of the statement will be denoted by $Y \not\longrightarrow (X_1, X_2)$. In this paper we show the following two results:

Theorem 2 *Let $K(3,1)$ be the non-convex 4-point configuration (see Fig. 2). Then for any configuration Y holds $Y \not\longrightarrow (K(3,1), K(3,1))$.*

Theorem 3 *Let T be the 3-point configuration and let K be a configuration such that at least two points of K lie inside the convex hull of K. Then for any configuration Y holds $Y \not\longrightarrow (T, K)$.*

Theorem 2 will be proved in Section 2 and Theorem 3 will be proved in Section 3. Before doing so we add a few remarks:

Remark. It follows from Ramsey's Theorem that for any two configurations C_1, C_2 in convex position there is a configuration Y with $Y \longrightarrow (C_1, C_2)$. (Any large

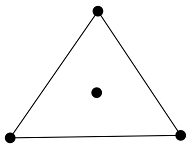

Figure 2: The configuration $K(3,1)$.

set in convex position will do.) On the other hand, any non-convex configuration contains a subconfiguration equivalent to $K(3,1)$. Therefore, Theorems 2 and 3 nearly characterize all pairs (K_1, K_2) of configurations for which there exists at least one configuration Y satisfying $Y \longrightarrow (K_1, K_2)$. The only remaining case is when K_1 is in convex position and K_2 is in convex position after removal of one point only.

Remark. This nearly complete solution is another indication that preserving the holes is a very restrictive condition for plane configurations. This should be compared with [NV98] where it has been proved that there are configurations X for which there is no Y such that $Y \longrightarrow (X, X)$ holds even for subconfigurations (which are not necessarily induced). This one can restate (in a slightly stronger form) as follows [NV98]: There exists a configuration X and a 2-coloring of all the pairs in the plane such that no subconfiguration equivalent to X is monochromatic.

The configuration X in [NV98] is finite but very large and in fact the proof given in [NV98] doesn't produce any bound on its size. On the other hand for many configurations X the existence of a configuration Y with $Y \longrightarrow (X, X)$ (for non-induced case) is guaranteed by [NV94] and our companion paper [NSV99].

2 Proof of Theorem 2

Let Y be a configuration. Choose a point p outside conv(Y) such that $Y \cup p$ is in general position. We color pairs of points of Y by red and blue so that a pair $\{a, b\} \in \binom{Y}{2}$ is colored red if the interior of the triangle abp contains an even number of points of Y, otherwise it is colored blue. We claim that no induced subconfiguration equivalent to $K(3,1)$ has all pairs of points colored by the same color.

Let K be an induced subcofiguration of Y equivalent to $K(3,1)$. We denote the points of K by a_1, a_2, a_3, a_4 in the order in which they are seen from p (see Fig. 3). It is easy to check that one of the 4-point configurations $\{p, a_1, a_2, a_3\}$, $\{p, a_2, a_3, a_4\}$ is in convex position and the other one is not in convex position. However, if $\{p, a_i, a_{i+1}, a_{i+2}\}$ is in convex position, then the interior of at least one of the triangles pa_ia_{i+1}, pa_ia_{i+2}, $pa_{i+1}a_{i+2}$ contains an even number of points of Y. Similarly, if $\{p, a_i, a_{i+1}, a_{i+2}\}$ is not in convex position, then the interior of at least one of the triangles pa_ia_{i+1}, pa_ia_{i+2}, $pa_{i+1}a_{i+2}$ contains an odd number of points of Y. It follows that the pairs $\{a_1, a_2\}$, $\{a_1, a_3\}$, $\{a_2, a_3\}$, $\{a_2, a_4\}$, $\{a_3, a_4\}$

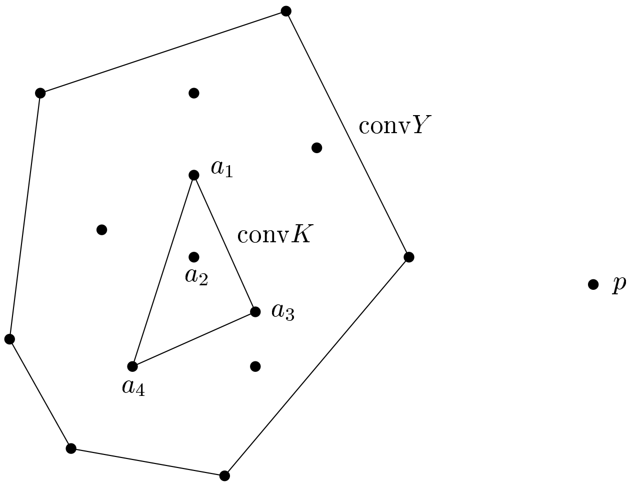

Figure 3: The points a_1 through a_4.

cannot be colored all by the same color. $\square$

3 Proof of Theorem 3

Our proof of Theorem 3 relies on the following lemma:

Lemma 4 *Let k be a configuration such that the convex hull of K contains at least two points of K in the interior. Then K contains an induced subconfiguration equivalent to one of the three configurations $K(3,2)$, $K(4,2)$, $K(3,3)$ shown in Fig. 4.*

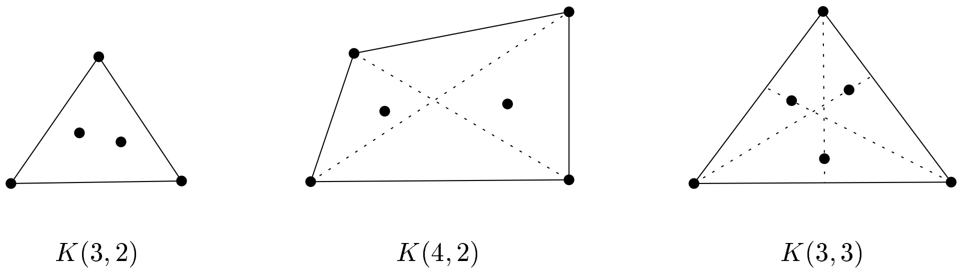

Figure 4: The configurations $K(3,2)$, $K(4,2)$, $K(3,3)$.

Proof of Lemma 4. First, observe that K contains an induced configuration K_0 such that conv K_0 has 3 or 4 vertices and contains at least two points of K in the interior. To see this, take a line through any pair of points of K inside conv K. It crosses the boundary of conv K in two different sides (segments) s_1 and s_2. Then we can take $K_0 = K \cap \mathrm{conv}(s_1 \cup s_2)$.

Now, consider the configuration K_0. If K_0 is equivalent to $K(4,2)$, then the lemma holds. Otherwise, if we take the convex hull of some three vertices of conv K_0, we obtain an induced subconfiguration K_1 such that conv K_1 is a triangle with at least two points of K inside it.

If there are more than three points inside conv K_1, then we can find three points in K_1 with fewer (but at least two) points of K in their convex hull as follows. Denote the vertices of conv K_1 by a, b, c, and let d be a point inside abc nearest to the line ab. The triangle dab contains no point inside. Thus, the interior of one of the triangles dac, dbc contains at least two points. Repeating this argument, we obtain an induced subconfiguration K_2 such that conv K_2 is a triangle with two or three points of K in the interior. If the interior of conv K_2 contains two points of K, then K_2 is equivalent to $K(3,2)$. So suppose that it contains three points of K.

Denote the vertices of conv K_2 by a, b, c, and let d be a point of K lying inside conv K_2. If the remaining two points of K inside conv K_2 lie in the same of the triangles abd, acd, bcd, then there is an induced configuration equivalent to $K(3,2)$. Thus, we may assume that the interior of abd contains a single point u of K and the interior of acd contains a single point v of K (see Fig. 5). If K_2 is equivalent to

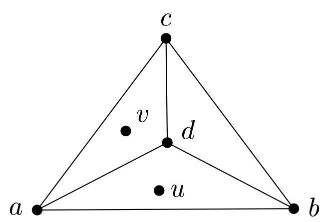

Figure 5: The points d, u, v.

$K(3,3)$, then the lemma holds. Otherwise, either the line cu separates the points b and d, or the line bv separates the points c and d. In the first case $K_2 \setminus \{b\}$ is equivalent to $K(3,2)$, in the second case $K_2 \setminus \{c\}$ is equivalent to $K(3,2)$. □

We now prove Theorem 3. Let K be a configuration such that at least two points of K lie inside conv K. Let Y be an arbitrary configuration. In the following, we define a 2-coloring of pairs of points of Y showing that $Y \not\longrightarrow (T, K)$.

We may assume that no pair of points of Y lies on a vertical line. For a pair of points $a, b, x(a) < x(b)$, we define S_{ab} as the set of points lying under the segment ab:

$$S_{ab} = \{p \in \mathbb{R}^2 : x(a) < x(p) < x(b) \text{ and } p \text{ lies below the line } ab\}.$$

We color a pair $\{a, b\}$ of points of Y red, if $|Y \cap S_{ab}| \equiv 1 \pmod 3$ (i.e., the number of points of Y in the strip S_{ab} is $1 \pmod 3$). Otherwise we color it blue. We call a configuration C *red*, if all pairs of points of C are colored red. Analogously we define *blue* configurations. It is easy to see that Y contains no empty three-point red configuration.

To show Theorem 3, it therefore suffices to show that Y contains no empty blue configuration equivalent to K. According to Lemma 4, Theorem 3 follows from the following lemma.

Lemma 5 *Let Y be a configuration with pairs of points colored as described above. Let $K_0 \subseteq Y$ be an empty blue subconfiguration of Y. Then*

(i) $K_0 \not\approx K(3,2)$,

(ii) $K_0 \not\approx K(4,2)$,

(iii) $K_0 \not\approx K(3,3)$.

We prove Lemma 5 in the following four sections.

4 Proof of Lemma 5(i)

Let K_0 be an empty subconfiguration of Y equivalent to $K(3,2)$. We will show that K_0 is not blue.

Let $a_1, a_2, \ldots, a_5$ be the points of K_0 listed according to increasing x-coordinate. We denote by S_1, S_2, S_3, S_4 the four strips (from left to right) determined by the lower envelope of $\operatorname{conv} K_0$ and by the 5 vertical halflines emanating from the points a_i downward (see Fig. 6). For $i = 1,2,3,4$, we define $c_i \in \{0,1,2\}$ by

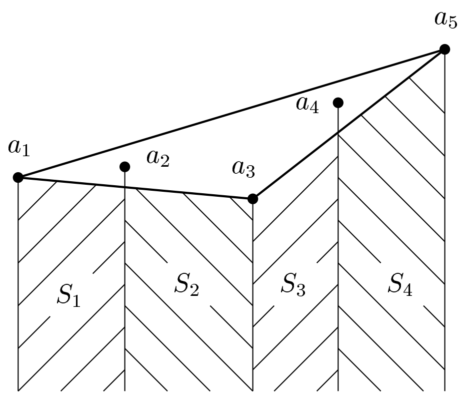

Figure 6: The strips S_i.

$c_i \equiv |(Y \setminus K_0) \cap S_i| (\mathrm{mod}\ 3)$. Further, we put $C = c_1 + c_2 + c_3 + c_4$.

Lemma 6 *If K_0 is blue then*

(i) $c_i = 0$ or $c_i = 2$, for every $i = 1,2,3,4$, and

(ii) either $C = 2$ or $C = 6$.

Proof. (i) If $c_i = 1$ for some i, then $\{a_i, a_{i+1}\}$ is red.

(ii) By (i), C is even.

If $C = 0$ (i.e., $c_1 = c_2 = c_3 = c_4 = 0$), choose i such that a_{i+1} lies below the line $a_i a_{i+2}$. Then the pair $\{a_i, a_{i+2}\}$ is colored red.

If $C = 8$ (i.e., $c_1 = c_2 = c_3 = c_4 = 2$), choose i such that a_{i+1} lies above the line $a_i a_{i+2}$. Then the pair $\{a_i, a_{i+2}\}$ is colored red.

Finally, if $C = 4$, then $\{a_1, a_5\}$ is colored red. □

We now suppose that K_0 is blue, and distinguish several cases.

Case 1: $\operatorname{conv} K_0 = \{a_1, a_2, a_5\}$ *and* a_2 *lies above the line* $a_1 a_5$. If $C = 2$, then either $c_1 = 0$ and $\{a_2, a_5\}$ is red, or $c_1 = 2$ and (at least) one of the pairs $\{a_2, a_4\}$, $\{a_3, a_5\}$ is red.

If $C = 6$, then either $c_1 = c_2 = 2$ and $\{a_1, a_3\}$ is red, or $c_1 + c_2 = c_3 = c_4 = 2$ and one of the pairs $\{a_1, a_4\}$, $\{a_3, a_5\}$ is red.

Case 2: $\operatorname{conv} K_0 = \{a_1, a_2, a_5\}$ *and* a_2 *lies below the line* $a_1 a_5$. If $C = 2$, then either $c_1 + c_2 = 0$ and $\{a_1, a_3\}$ is red, or $c_1 + c_2 = 2$ and (at least) one of the pairs $\{a_1, a_4\}$, $\{a_3, a_5\}$ is red.

If $C = 6$, then either $c_1 = 2$ and $\{a_2, a_5\}$ is red, or $c_1 = 0$ and one of the pairs $\{a_2, a_4\}$, $\{a_3, a_5\}$ is red.

Case 3: $\operatorname{conv} K_0 = \{a_1, a_3, a_5\}$ *and* a_3 *lies above the line* $a_1 a_5$. If $C = 2$, then either $c_1 + c_2 = 0$ and $\{a_1, a_3\}$ is red, or $c_3 + c_4 = 0$ and $\{a_3, a_5\}$ is red.

If $C = 6$, then either $c_2 + c_3 = 4$ and $\{a_2, a_4\}$ is red, or $c_1 = c_2 + c_3 = c_4 = 2$ and one of the pairs $\{a_1, a_4\}$, $\{a_2, a_5\}$ is red.

Case 4: $\operatorname{conv} K_0 = \{a_1, a_3, a_5\}$ *and* a_3 *lies below the line* $a_1 a_5$. If $C = 2$, then either $c_2 + c_3 = 0$ and $\{a_2, a_4\}$ is red, or $c_2 + c_3 = 2$ and one of the pairs $\{a_1, a_4\}$, $\{a_2, a_5\}$ is red.

If $C = 6$, then either $c_1 + c_2 = 4$ and $\{a_1, a_3\}$ is red, or $c_3 + c_4 = 4$ and $\{a_3, a_5\}$ is red.

Case 5: $\operatorname{conv} K_0 = \{a_1, a_4, a_5\}$. This case is always symmetric to Case 1 or 2.
□

5 Configuration $K(3, 1)$

In this section we prove a lemma which we apply in the proof of Lemma 5(ii),(iii). Let $K(3, 1)$ be the non-convex 4-point configuration (see Fig. 2), and let K_1 be an empty blue subconfiguration of Y equivalent to $K(3, 1)$. Let a_1, a_2, a_3, a_4 be the points of K_1 listed according to increasing x-coordinate. Similarly as in the previous section, let S_1, S_2, S_3 be the three strips bounded by the lower envelope of $\operatorname{conv} K_1$ and by the four vertical halflines emanating from the points a_i downward. Define $c_i \in \{0, 1, 2\}$ by $c_i \equiv |(Y \setminus K_1) \cap S_i| (\bmod 3)$ and put $C = c_1 + c_2 + c_3$.

Lemma 7 *Let* K_1 *be a configuration as above. Then*

(i) if a_2, a_3 *lie below the line* $a_1 a_4$ *then* $C = 4$*, and*

(ii) if a_2, a_3 *lie above the line* $a_1 a_4$ *then* $C = 2$.

Proof. Certainly, each c_i is 0 or 2.

If $C = 0$ then either $\{a_1, a_3\}$ or $\{a_2, a_4\}$ is red.

If $C = 6$ then also either $\{a_1, a_3\}$ or $\{a_2, a_4\}$ is red.

In case (i), if $C = 2$ then $\{a_1, a_4\}$ is red.

In case (ii), if $C = 4$ then $\{a_1, a_4\}$ is red.

This completes the proof. $\square$

6 Proof of Lemma 5(ii)

Let K_0 be an empty blue subconfiguration of Y equivalent to $K(4, 2)$. Let a, b, c, d be the vertices of $\operatorname{conv} K_0$ listed in the counterclockwise order so that a is the leftmost of them. We say that a point p *lies under* a segment qr if $x(q) < x(p) < x(r)$ and p lies strictly under the line qr. We distinguish three cases.

Case 1: d is the rightmost point of K_0. By Lemma 7(i) for the configuration $Y \cap \operatorname{conv}\{a, b, c\}$, $0 (\mathrm{mod}\ 3)$ points of Y lie under the segment ac. By Lemma 7(i) for the configuration $Y \cap \operatorname{conv}\{a, c, d\}$, $0 (\mathrm{mod}\ 3)$ points of Y lie under the segment ad. Since there is one point of Y inside the triangle acd, it follows that $1 (\mathrm{mod}\ 3)$ points of Y lie under the segment cd. Thus $\{c, d\}$ is red.

Case 2: c is the rightmost point of K_0. By Lemma 7(i) for $Y \cap \operatorname{conv}\{a, b, c\}$, $0 (\mathrm{mod}\ 3)$ points of Y lie under the segment ac. By Lemma 7(ii) for $Y \cap \operatorname{conv}\{a, c, d\}$, $2 (\mathrm{mod}\ 3)$ points of Y lie under the segment ac – a contradiction.

Case 3: b is the rightmost point of K_0. By Lemma 7(ii) for $Y \cap \operatorname{conv}\{a, c, d\}$, $2 (\mathrm{mod}\ 3)$ points of Y lie under the segment ac. By Lemma 7(ii) for $Y \cap \operatorname{conv}\{a, b, c\}$, $2 (\mathrm{mod}\ 3)$ points of Y lie under the segment ab. Since there is one point of Y inside the triangle abc, it follows that $1 (\mathrm{mod}\ 3)$ points of Y lie under the segment bc. Thus $\{b, c\}$ is red. $\square$

7 Proof of Lemma 5(iii)

Let K_0 be an empty blue subconfiguration of Y equivalent to $K(3, 3)$. Let a, b, c be the vertices of $\operatorname{conv} K_0$ denoted so that $x(a) < x(c) < x(b)$, and let a', b', c' be the remaining three points of K_0 denoted uniquely so that the line cc' separates the points a, b' from the points a', b (see Fig. 7). Without loss of generality, we may

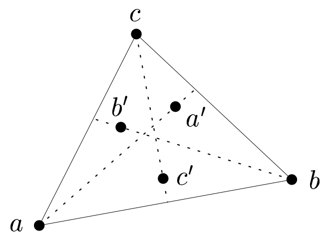

Figure 7: The points a', b', c'.

assume that $x(c') < x(c)$. We define strips $S_1, S_2, \ldots, S_5$ and numbers $c_1, c_2, \ldots, c_5$ and $C = \sum c_i$ similarly as in Sections 4 and 6. We distinguish four cases.

Case 1: c lies below the line ab and $x(b') < x(c')$. By Lemma 7(i) for $\{b, b', c, a'\}$, $C - c_1 \equiv 1 (\mathrm{mod}\ 3)$.

Similarly, by Lemma 7(i) for $\{b, c, c', a'\}$, $C - c_1 - c_2 \equiv 1 (\mathrm{mod}\ 3)$.

Thus, $c_2 = 0$. Consequently, $c_1 = 2$ (otherwise $c_1 = 0$ and $\{a, c'\}$ is red). Thus, $C \equiv 0 (\mathrm{mod}\ 3)$ and $\{a, b\}$ is red.

Case 2: c lies below the line ab and $x(c') < x(b')$. By Lemma 7(i) for $\{b, c, c', a'\}$,

$$C - c_1 + 1 \equiv 1 (\mathrm{mod}\ 3).$$

If $c_1 = 0$, then $C \equiv 0 (\mathrm{mod}\ 3)$ and $\{a, b\}$ is red.

If $c_1 = 2$, then $c_2 = 0$ (otherwise $\{a, b'\}$ is red) and $C - c_1 - c_2 \equiv 0 (\mathrm{mod}\ 3)$, but $C - c_1 - c_2 \equiv 1 (\mathrm{mod}\ 3)$ by Lemma 7(i) for $\{b, b', c, a'\}$ – a contradiction.

Case 3: c lies above the line ab and $x(b') < x(c')$. By Lemma 7(i) for $\{b, b', c, a'\}$, $C - c_1 + 1 \equiv 2 (\mathrm{mod}\ 3)$.

By Lemma 7(i) for $\{b, c, c', a'\}$, $C - c_1 - c_2 \equiv 2 (\mathrm{mod}\ 3)$.

Thus, $c_2 = 2$. Consequently, $c_1 = 0$ since otherwise $\{a, c'\}$ is red. It follows that $C \equiv 1 (\mathrm{mod}\ 3)$ and $\{a, b\}$ is red.

Case 4: c lies above the line ab and $x(c') < x(b')$. By Lemma 7(i) for $\{b, b', c, a'\}$,

$$C - c_1 - c_2 \equiv 2 (\mathrm{mod}\ 3).$$

By Lemma 7(i) for $\{b, c, c', a'\}$,

$$C - c_1 \equiv 2 (\mathrm{mod}\ 3).$$

Thus, $c_2 = 0$. Consequently, $c_1 = 2$ (otherwise $c_1 = 1$ or $\{a, b'\}$ is red). Thus, $C \equiv 1 (\mathrm{mod}\ 3)$ and $\{a, b\}$ is red. □

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