Ramsey type theorems for tournaments and for H-free graphs

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Abstract

A graph is called H-free if it contains no induced copy of H. We discuss the following question raised by Erdős and Hajnal. Is it true that for every graph H, there exists an $\varepsilon(H) > 0$ such that any H-free graph with n vertices contains either a complete or an empty subgraph of size at least $n^{\varepsilon(H)}$? We answer this question in the affirmative for a special class of graphs, and give an equivalent reformulation for tournaments. In order to prove the equivalence we obtain several Ramsey type results for tournaments.

1 Introduction

Given a graph G with vertex set $V(G)$ and edge set $E(G)$, let $\alpha(G)$ and $\omega(G)$ denote the size of the largest independent set (empty subgraph) and the size of the largest clique (complete subgraph) in G, respectively. A subset $U \subseteq V(G)$ is called *homogeneous*, if it is either an independent set or a clique. Denote by $\hom(G)$ the size of the largest homogeneous set in G, i.e., let

$$\hom(G) = \max\left(\alpha(G), \omega(G)\right).$$

If H is not an induced subgraph of G, then we say that G is an *H-free* graph.

According to classical Ramsey theory, $\hom(G) \geq \frac{1}{2}\log_2 n$ for every graph G with n vertices [ES35], and there exists some G with $\hom(G) < 2\log_2 n$ (see [E47]). Erdős and Hajnal [EH89] raised the possibility that "the following could be true."

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Conjecture 1 *For every graph H, there exists a positive $\varepsilon = \varepsilon(H)$ such that every H-free graph with n vertices has a homogeneous set whose size is at least n^{ε}.*

Erdős and Hajnal confirmed their conjecture for every graph H which belongs to the class $\mathcal{H}$ defined recursively, as follows:

(i) K_1, the graph consisting of a single vertex, belongs to $\mathcal{H}$;

(ii) if H_1 and H_2 are two vertex disjoint graphs that belong to $\mathcal{H}$, then their disjoint union as well as the graph obtained from this union by connecting every vertex of H_1 to every vertex of H_2, also belong to $\mathcal{H}$.

Gyárfás [G97] noticed that it follows from a well known result of Seinsche [Se74] that Conjecture 1 is also true for all graphs generated by the above rules starting with P_4, a simple path with 4 vertices, and K_1.

Our first theorem extends both of these results. If Conjecture 1 is true for some graph H, then we say that H has the *Erdős-Hajnal property.*

For any graph H with vertex set $V(H) = \{v_1, \ldots, v_k\}$ and for any other graphs, $F_1, \ldots, F_k$, let $H(F_1, \ldots, F_k)$ denote the graph obtained from H by replacing each v_i with a copy of F_i, and joining a vertex of the copy of F_i to a vertex of a copy of F_j, $j \neq i$, if and only if $v_i v_j \in E(H)$. The copies of F_i, $i = 1, \ldots, k$, are supposed to be vertex disjoint.

Theorem 1.1 *If $H, F_1, \ldots, F_k$ have the Erdős-Hajnal property, then so does $H(F_1, \ldots, F_k)$.*

In other words, the Erdős-Hajnal property is preserved by replacement. This enables us to verify that Conjecture 1 is true, e.g., for the graphs depicted in Figure 1, which answers some questions of Gyárfás [G97].

No non-perfect graph is known to have the Erdős-Hajnal property. Unfortunately, in this respect Theorem 1.1 cannot offer any help. Indeed, according to a result of Lovász [L83], which played a key role in his proof of the Weak Perfect Graph Conjecture [L72], perfectness is also preserved by replacement. It is an outstanding open problem to decide whether the smallest non-perfect graph, the cycle of length 5, has the Erdős-Hajnal property. As Lovász pointed out, there is an even simpler unsolved

Problem *Does there exist a positive constant ε so that for every graph G on n vertices such that neither G nor its complement $\bar{G}$ contains an induced odd cycle whose length is at least 5, $\hom(G) \geq n^{\varepsilon}$?*

It is easy to formulate analogous questions for tournaments. A tournament with no directed cycle is called *transitive.* If a tournament has no subtournament isomorphic to T, then it is called *T-free.*

It is well known [EEH73],[S74] that every tournament of n vertices contains a transitive subtournament whose size is at least $c \log n$, and that this result is tight apart from the value of the constant.

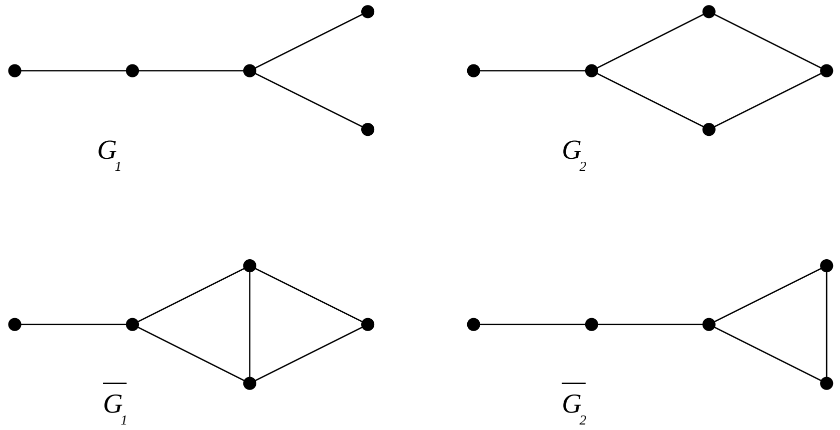

Figure 1: G_1 and G_2 have the Erdős-Hajnal property

Conjecture 2 *For every tournament T, there exists a positive $\varepsilon = \varepsilon(T)$ such that every T-free tournament with n vertices has a transitive subtournament whose size is at least n^{ε}.*

Theorem 1.2 *Conjecture 1 and Conjecture 2 are equivalent.*

In order to prove Theorem 1.2 we establish a certain Ramsey theoretic result for tournaments, which is interesting in its own right. A tournament T with a linear order $<$ on its vertex set is called an *ordered tournament* and is denoted by $(T, <)$. An ordered tournament $(T, <)$ is said to be an *induced subtournament* of another one, $(T', <')$, if there is a function $f : V(T) \to V(T')$ satisfying the conditions

(i) $f(u) <' f(v)$ if and only if $u < v$,

(ii) $\overrightarrow{f(u)f(v)} \in E(T')$ if and only if $\vec{uv} \in E(H)$.

Theorem 1.3 *For any ordered tournament $(T, <)$, there exists a tournament T' such that, for every ordering $<'$ of T', $(T, <)$ is an induced subtournament of $(T', <')$.*
Moreover, if T has n vertices, there exists a T' with the required property, whose number of vertices is at most $O\left(n^3 \log^2 n\right)$.

We further show that the $O(n^3 \log^2 n)$ estimate is not very far from being tight. In fact, if $(T, <)$ is any tournament on n vertices and T' satisfies the condition above for T, then T' must have at least $\Omega(n^2)$ vertices.

By choosing a bigger tournament T', one can ensure a single tournament that contains all ordered tournaments on n vertices, in any ordering. Specifically, we prove the following.

Theorem 1.4 *Given an integer N, let n_0 be the largest integer such that*

$$\binom{N}{n_0} 2^{-\binom{n_0}{2}} \geq 1,$$

and put $n = n_0 - 2$. Then, for all sufficiently large N, there exists an ordered tournament T' on N vertices such that in any ordering it contains every ordered tournament on n vertices.

Note that the above estimate for n is clearly tight, up to an *additive* error of 2. A similar statement holds for induced subgraphs, improving a result in [BLTV99].

The rest of this paper is organized as follows. Theorem 1.1 is proved in Section 2. The proofs of Theorems 1.3 and 1.4 appear in Section 3. Section 4 contains the proof of Theorem 1.2.

2 Graphs with the Erdős-Hajnal property

In this section we prove Theorem 1.1. Obviously, it is sufficient to show the following weaker version of the theorem.

Theorem 2.1 *Let H and F be graphs having the Erdős-Hajnal property, $V(H) = \{v_1, v_2, \ldots, v_k\}$. Then the graph $H(F, v_2, \ldots, v_k)$, obtained by replacing v_1 with H, also has this property.*

Proof: Let H_0 denote the graph obtained from H by the deletion of v_1. For simplicity, write $H(F)$ for $H(F, v_2, \ldots, v_k)$. Let G be an $H(F)$-free graph with n vertices, and assume that $\hom(G) < n^{\varepsilon(H)\delta}$. We would like to get a contradiction, provided that $\delta > 0$ is sufficiently small.

Let $m := \lceil n^\delta \rceil$. By the definition of $\varepsilon(H)$, any m-element subset of $U \subset V(G)$ must induce at least one subgraph isomorphic to H. Otherwise, we would find a homogeneous subset of $m^{\varepsilon(H)} > \hom(G)$ in the subgraph of G induced by U, which is impossible. Therefore, G has at least $\binom{n}{m}/\binom{n-k}{m-k}$ induced subgraphs isomorphic to H.

Since the number of induced subgraphs of G isomorphic to H_0 cannot exceed $\binom{n}{k-1}$, there exists an induced subgraph isomorphic to H_0, which can be extended to an induced subgraph isomorphic to H in at least

$$M := \frac{\binom{n}{m}}{\binom{n-k}{m-k}\binom{n}{k-1}} \tag{1}$$

different ways. In other words, there is a $k-1$ element subset $V_0 \subset V(G)$ and an M-element subset $W \subset V(G)$ such that, for every $w \in W$, $V_0 \cup \{w\}$ induces a subgraph of G isomorphic to H. The elements of W can be connected to the vertices in V_0 in at most $\binom{k-1}{\lfloor (k-1)/2 \rfloor}$ different ways. Let $\bar{W}$ be a subset of W such that

$$|\bar{W}| \geq \frac{\binom{n}{m}}{\binom{n-k}{m-k}\binom{n}{k-1}\binom{k-1}{\lfloor (k-1)/2 \rfloor}}, \tag{2}$$

and any element of $\bar{W}$ is connected to exactly the same elements of V_0.

Consider now the subgraph $G|_{\bar{W}}$ of G induced by $\bar{W}$. This graph must be F-free, otherwise G would not be $H(F)$-free. Since F has the Erdős-Hajnal property, we know that

$$\hom(G|_{\bar{W}}) \geq |\bar{W}|^{\varepsilon(F)}.$$

On the other hand,

$$n^{\varepsilon(H)\delta} > \hom(G) \geq \hom(G|_{\bar{W}}).$$

Comparing the last two inequalities and plugging in our estimate (2) for $|\bar{W}|$, we obtain that

$$n^{\delta\varepsilon(H)/\varepsilon(F)} > \frac{\binom{n}{m}}{\binom{n-k}{m-k}\binom{n}{k-1}\binom{k-1}{\lfloor (k-1)/2 \rfloor}} > n^{1-k\delta},$$

which gives the desired contradiction, provided that

$$\delta < \frac{\varepsilon(F)}{\varepsilon(H)+k\varepsilon(F)}.$$

□

3 Ramsey type theorems for tournaments

The proof of Theorem 1.3 is probabilistic. We need the following lemma.

Lemma 3.1 *Let $t > n > 1$ be two positive integers, and let $S = (a_1, a_2, \ldots, a_{tn})$ be a sequence of length tn. Let $g : S \mapsto R$, where $R = \{1, 2, \ldots, t\}$ be a function, and suppose that for each $p \in R$, $|\{i : g(a_i) = p\}| = n$. Let $f : S \mapsto N$, where $N = \{1, 2, \ldots, n\}$ be a random function, obtained by choosing, for each element $a_i \in S$, randomly and independently, the value of $f(a_i) \in N$ according to a uniform distribution. Let E be the following event; there exist $1 \leq i_1 < i_2 < \ldots < i_n \leq nt$ such that $g(a_{i_j}) \neq g(a_{i_k})$ for all $1 \leq j < k \leq n$, and $f(a_{i_j}) = j$ for all $1 \leq j \leq n$. Then the probability that E does not hold is at most*

$$\sum_{q=0}^{n-1} \binom{tn}{q} \frac{n^{q(n-1)}(n-1)^{tn-nq}}{n^{tn}} \leq (\frac{4et}{n})^n e^{-t}.$$

Proof : To estimate the number of functions f for which the event E fails we argue as follows. Given such an f, let i_1 be the smallest integer (if it exists) such that $f(a_{i_1}) = 1$. Assuming $i_1 < i_2 < \ldots < i_{j-1}$ have already been defined, and assuming that $f(i_s) = s$ for all $s < j$ and that the elements $g(a_{i_s}), s < j$ are pairwise distinct, let i_j be the smallest integer (if it exists) satisfying $i_j > i_{j-1}$, $f(a_{i_j}) = j$ and $g(a_{i_j}) \neq g(a_{i_s})$ for all $s < j$. Note that since the event E fails, this process must terminate after some $q \leq n-1$ elements i_s have been defined. Note, now, that if k is an index satisfying $i_{s-1} < k < i_s$, and $g(a_k)$ differs from $g(a_{i_j})$ for all $j \leq q$ (or even just for all $j < s$), then $f(a_k)$ cannot be equal to s (since otherwise we would have defined $i_s = k$). Since there is a similar restriction for the value of $f(a_k)$ for $k < i_1$ and for $k > i_q$, it follows that once the sequence $i_1 < i_2 \ldots < i_q$ have been defined, the value of $f(a_k)$ can attain at most $n-1$ values for all but at most $tn - nq$ elements a_k. Therefore, the total number of functions f for which the event E fails is at most

$$\sum_{q=0}^{n-1} \binom{tn}{q} n^{q(n-1)}(n-1)^{tn-nq}.$$

Since the total number of possible functions f is n^{tn}, the probability that E does not hold is at most

$$\sum_{q=0}^{n-1} \binom{tn}{q} \frac{n^{q(n-1)}(n-1)^{tn-nq}}{n^{tn}} \leq \sum_{q=0}^{n-1} (\frac{etn}{q})^q (\frac{n^n}{(n-1)^n})^q \frac{1}{n^q}(1-\frac{1}{n})^{tn}$$

$$\leq \left(\frac{et}{n}(1+\frac{1}{n-1})^n\right)^n e^{-t} \leq (\frac{4et}{n})^n e^{-t}.$$

$\square$.

Proof of Theorem 1.3: Let $(T, <)$ be an ordered tournament on the set $N = \{1, 2, \ldots, n\}$ of n vertices, ordered naturally. We may and will assume that n is sufficiently large. Let $c > 3$ be an absolute constant, and let t be the smallest integer satisfying $t > cn \log n$ such that $t-1$ is a prime. By the known estimates for the distribution of primes, $t = (1+o(1))cn \log n$. Let P be a projective plane of order $t-1$. Each line of P contains precisely t points, and the number of points in P is $(t-1)^2 + t < t^2$. Replace each point $p \in P$ by a set S_p of n points, where all sets S_p are pairwise disjoint. Construct a tournament T' on the set $\cup_{p \in P} S_p$ of less than nt^2 vertices as follows. For every line l in P, let $f_l : \cup_{p \in l} S_p \mapsto N = \{1, 2, \ldots, n\}$ be a random function, where each image $f_l(u)$ is chosen, randomly, uniformly and independently in N, and the functions corresponding to different lines are chosen independently. For $u, v \in \cup_{p \in l} S_p$, where $u \in S_p,\ v \in S_{p'}$ and $p \neq p'$, let $\vec{uv}$ be a directed edge if and only if $\overrightarrow{f_l(u) f_l(v)}$ is a directed edge of T. The edges with two endpoints in the same set S_p are oriented arbitrarily.

To complete the proof we show that almost surely (that is, with probability that tends to 1 as n tends to infinity), T' contains an ordered copy of T in any ordering. Fix an ordering $<'$ of T', and let us estimate the probability that in this ordering $(T', <')$ contains no ordered copy of T. For each line l in the projective plane, the ordering $<'$ induces an ordering of the nt vertices $\cup_{p \in l} S_p$. Let $S = (a_1, a_2, \ldots, a_{tn})$ be this induced ordering. Define $g(a_i) = p$ if $a_i \in S_p$. Then, for every $p \in l$, $|\{i : g(a_i) = p\}| = |S_p| = n$. Observe, now, that by Lemma 3.1, the probability that $(T, <)$ is not an induced subtournament of the induced ordered subgraph of $(T', <')$ on S, using only the edges between distinct groups S_p, $p \in l$, is at most $(\frac{4et}{n})^n e^{-t}$. This is because if the event E in the formulation of the lemma holds for $f = f_l$, then the vertices $a_{i_1}, \ldots, a_{i_n}$ supply an induced copy of T as required. Since the events for distinct lines are totally independent, the probability that $(T', <')$ contains no ordered copy of $(T, <)$ is at most

$$((\frac{4et}{n})^n e^{-t})^{(t-1)^2+t} = e^{-(1+o(1))c^3 n^3 \log^3 n}.$$

The total number of orderings of T' is $(n((t-1)^2+t))! \leq e^{(1+o(1))3c^2 \log^3 n}$, and as $c > 3$, by our choice, the probability that T' fails to contain a copy of T in some ordering is $o(1)$, completing the proof. $\Box$

We next show that the $O(n^3 \log^2 n)$ upper bound cannot be replaced by $o(n^2)$. We need the following known fact. For completeness, we include a short proof.

Lemma 3.2 *There exists an absolute constant C such that the number $|Aut(T)|$ of automorphisms of any tournament T on n vertices does not exceed C^n.*

Proof: Let $F(n)$ denote the maximum possible number of automorphisms of a tournament on n vertices. Without trying to optimize the constant C we show that $F(n) \leq \frac{6^n}{n+1}$ for every $n \geq 1$. We apply induction on n, the result being trivial for $n = 1$. Let T be a tournament on n vertices. It is easy to see that it contains a vertex v whose outdegree and indegree are both at least $(n-2)/4$. Indeed, either it contains a set X of at least $n/2$ vertices of outdegree at least $(n-1)/2$, or it contains a set Y of at least $n/2$ vertices of indegree at least $(n-1)/2$. In the former case let v be the vertex of maximum indegree in the induced subtournament on X, and in the latter, let v be the vertex of maximum outdegree is in the induced subtournament on Y.

There are n possible images for v in an automorphism of T, and once its image u is fixed, its out-neighbors have to be mapped isomorphically to the out-neighbors of u and its in-neighbors have to be mapped isomorphically to the in-neighbors of u. This gives, denoting the number of out-neighbors of v by r,

$$F(n) \leq n \cdot max_{(n-2)/4 \leq r \leq (3n-2)/4} F(r)F(n-1-r).$$

By the induction hypothesis the right hand side is at most

$$n \cdot max_{(n-2)/4 \leq r \leq (3n-2)/4} \frac{6^r}{r+1} \frac{6^{n-r-1}}{n-r} \leq \frac{6^n}{n+1}.$$

$\Box$

Theorem 3.3 *There exists an absolute constant b with the following property. Let $(T, <)$ be an ordered tournament on n vertices, and suppose T' is another tournament such that for every ordering $<'$ of T', $(T, <)$ is an induced subtournament of T. Then T' has at least bn^2 vertices.*

Proof: Let N be the number of vertices of T'. Then the total number of induced (not necessarily ordered) copies of T in T' is at most $\binom{N}{n}|Aut(T)|$ which, by Lemma 3.2 and its proof, does not exceed $(\frac{eN}{n})^n 6^n$. It follows that the probability that for a random ordering $<'$ of T' at least one of these copies is ordered, is at most

$$(\frac{eN}{n})^n 6^n \frac{1}{n!} \leq (\frac{6e^2 N}{n^2})^n.$$

If $N < n^2/6e^2$ this number is less than 1, implying that there is an ordering $<'$ with no ordered copy of $(T,<)$. Thus $N \geq n^2/6e^2$, completing the proof. $\Box$

The discussion for tournaments can be easily repeated for induced subgraphs in graphs. A simple undirected graph H with a linear order $<$ on its vertex set is called an *ordered graph* and is denoted by $(H,<)$. An ordered graph $(H,<)$ is said to be an *induced subgraph* of another one, $(H',<')$, if there is a function $f : V(H) \to V(H')$ such that, for any $u, v \in V(H)$,

(i) $f(u) <' f(v)$ if and only if $u < v$,

(ii) $f(u)f(v) \in E(H')$ if and only if $uv \in E(H)$.

Repeating the proof of Theorem 1.3 for ordered graphs, we obtain

Theorem 3.4 *For any ordered graph $(H,<)$, there exists a graph H' such that, for every ordering $<'$ of H', $(H,<)$ is an induced subgraph of $(H',<')$.*
Moreover, if H has n vertices, there exists an H' with the required property, whose number of vertices is at most $O\left(n^3 \log^2 n\right)$. $\Box$

A qualitative version of this result can also be deduced from the results of Ginn [G99] and Greenwell, Hemminger and Kleirlein [GHK73]. Note that there is no nontrivial analog to Theorem 3.3 here, since the number of automorphisms of undirected graphs on n vertices can be as large as $n!$ In fact, if $(H,<)$ is an ordered complete graph on n vertices, then the graph $H' = H$ has only n vertices and contains an induced ordered copy of $(H,<)$ in any ordering.
Returning to tournaments, we next describe the proof of Theorem 1.4.
Proof of Theorem 1.4: Let N, n_0 and n be as in the statement of the theorem, and let T' be a random tournament on the N vertices $1, 2, \ldots, N$ obtained by choosing, for each pair of vertices i, j of T, randomly, uniformly and independently, either the edge $\vec{ij}$ or the edge $\vec{ji}$. We assume, whenever this is needed, that N is sufficiently large. To complete the proof we show that almost surely T contains, in every ordering, an ordered copy of any tournament on n vertices. To this end, fix an ordering $<'$ of T', and fix an ordered tournament T on n vertices. We estimate the probability that in this ordering T' contains no ordered copy of T, using the (generalized) Janson Inequality (c.f., e.g., [AS92], p. 96). A similar proof can be given using martingales or an inequality of Talagrand. The computation here is very similar to the one estimating the probability that the clique number of the random graph $G(n, 1/2)$ is smaller by more than 2 than its expected value.

For each set K of n vertices of T', let B_K be the event that the induced subgraph of $(T', <')$ on K is an ordered copy of $(T, <)$. Then the probability $Pr(B_K)$ of each event B_K is precisely $2^{-\binom{n}{2}}$. Define $\mu = \binom{N}{n} 2^{-\binom{n}{2}}$ and note that this is the expected number of ordered copies of $(T, <)$ in $(T', <')$. A simple computation shows that the number n_0 defined in the statement of the theorem satisfies $n_0 = (1 + o(1)) 2 \log_2 N$, implying that for the function $f(m) = \binom{N}{m} 2^{-\binom{m}{2}}$ and for each m close to n_0, $f(m+1)/f(m) = N^{-1+o(1)}$. Therefore, $\mu \geq N^{2-o(1)}$ and the probability of each event B_K is $o(1)$.

For two subsets K and K', each containing n vertices of T', let $K \sim K'$ denote the fact that $2 \leq |K \cap K'| \leq n-1$. Define, further, $\Delta = \sum_{K \sim K'} Pr(B_K \wedge B_{K'})$, where the sum ranges over all ordered pairs K, K' as above. By the generalized Janosn Inequality, the probability that none of the events B_k holds (that is, the probability that $(T', <')$ contains no ordered copy of $(T, <)$) is at most

$$e^{-(1+o(1))\mu^2/2\Delta}, \tag{3}$$

provided $\Delta \geq \mu$. If $\Delta \leq \mu$ then, by the Janson Inequality (c.f. [AS92], p. 96), this probability is at most

$$e^{-(1/2-o(1))\mu} \leq e^{-\Omega(N^2)}.$$

The technical part of the proof is a careful estimate of the quantity Δ/μ^2. Observe that if $|K \cap K'| = i$ (≥ 2) then $Pr(B_K \wedge B_{K'}) \leq 2^{-2\binom{n}{2}+\binom{i}{2}}$. In fact, either this probability is zero, or it is precisely the above expression. It thus follows that $\Delta \leq \sum_{i=2}^{n-1} \Delta_i$, where

$$\Delta_i = \binom{N}{n}\binom{n}{i}\binom{N-n}{n-i} 2^{-2\binom{n}{2}+\binom{i}{2}}.$$

Therefore,

$$\frac{\Delta_i}{\mu^2} = \frac{\binom{N}{n}\binom{n}{i}\binom{N-n}{n-i} 2^{-2\binom{n}{2}+\binom{i}{2}}}{\binom{N}{n}^2 2^{-2\binom{n}{2}}}$$

$$= \frac{\binom{n}{i}\binom{N-n}{n-i} 2^{\binom{i}{2}}}{\binom{N}{n}} \leq (\frac{n}{N})^i \binom{n}{i} 2^{\binom{i}{2}} \leq \left(\frac{n^2}{N} 2^{(i-1)/2}\right)^i.$$

It follows that

$$\frac{\Delta_2}{\mu^2} \leq O(\frac{n^4}{N^2}) \tag{4}$$

and that for each i satisfying, say, $3 \leq i \leq 100$,

$$\frac{\Delta_i}{\mu^2} \leq O(\frac{n^6}{N^3}). \tag{5}$$

Furthermore, for $100 < i \leq 1.9\log_2 N$

$$\frac{\Delta_i}{\mu^2} \leq (\frac{n^2}{N^{0.05}})^i < \frac{1}{N^5}. \tag{6}$$

For each i in the range $1.9\log_2 N \leq i \leq n-1$ put $i = n-j$, and note that $1 \leq j \leq (1+o(1))0.1\log_2 N$ and

$$\frac{\Delta_i}{\mu^2} \leq \frac{1}{N^{2-o(1)}}\frac{\Delta_i}{\mu} = \frac{1}{N^{2-o(1)}}\frac{\binom{N}{n}\binom{n}{j}\binom{N-n}{j}2^{-\binom{n}{2}-j(n-j)-\binom{j}{2}}}{\binom{N}{n}2^{-\binom{n}{2}}}$$

$$\leq \frac{1}{N^{2-o(1)}}\binom{n}{j}\binom{N-n}{j}2^{-j(n-j)} \leq \frac{1}{N^{2-o(1)}}\left(nN2^{-(n-j)}\right)^j$$

$$\leq \frac{1}{N^{2-o(1)}}\left(\frac{n}{N^{0.9-o(1)}}\right)^j \leq \frac{1}{N^{2.9-o(1)}}.$$

Combining the last inequality with inequalities (4), (5) and (6), we conclude that $\frac{\Delta}{\mu^2} \leq O(\frac{n^4}{N^2})$.

Substituting in (3) we conclude that the probability that $(T', <')$ contains no ordered copy of $(T, <)$ is at most $e^{-\Omega(N^2/n^8)}$, provided $\Delta \geq \mu$. If $\Delta \leq \mu$, this probability can be bounded by an even smaller quantity, $e^{-\Omega(N^2)}$, as mentioned in the paragraph following (3).
Since the total number of orderings of T' is less than $N^N = e^{N\log N}$ and the total number of tournaments T on n vertices is $2^{\binom{n}{2}}$ we conclude that the probability that $(T', <')$ fails to contain some tournament of size n in some ordering is at most

$$e^{N\log N}2^{\binom{n}{2}}e^{-\Omega(N^2/n^8)} = o(1),$$

completing the proof. $\square$

Repeating the above proof for graphs instead of tournaments we obtain the following.

Theorem 3.5 *Given an integer N, let n_0 be the largest integer such that*

$$\binom{N}{n_0}2^{-\binom{n_0}{2}} \geq 1,$$

and put $n = n_0 - 2$. Then, for all sufficiently large N, the following holds almost surely. The random graph $G(N, 1/2)$ contains, in any ordering, an induced copy of every ordered graph on at most n vertices.

This improves significantly a result in [BLTV99], where the above statement is proved for induced copies of every ordered graph of size at most $(1 + o(1))\sqrt{n}$.

4 Tournaments and H-free graphs

In this section we prove Theorem 1.2. We need the following well known

Lemma 4.1 ([ES35]) *For any two total orderings of the same (k^2+1)-element set V, there is a $(k+1)$-element subset $U \subseteq V$ such that either the order of any two elements of U is the same, or the order of any two elements is opposite in the two orderings.* $\square$

We say that a tournament T has the *Erdős-Hajnal property* if there exists a positive $\varepsilon = \varepsilon(T)$ such that every T-free tournament with n vertices has a transitive subtournament whose size is at least n^{ε}.

To any tournament T and to any ordering $<$ of its vertex set, assign an ordered graph $(H(T), <)$ on the same vertex set, as follows. Join two vertices $u < v$ by an edge of $H(T)$ if and only if the edge connecting them in T was directed towards v. Similarly, assign to any ordered graph $(H, <)$ an ordered tournament $(T(H), <)$ with the same vertex set, by connecting $u < v$ with an edge directed towards v if $uv \in E(H)$ and with an edge directed towards u if $uv \neq E(H)$.

Now we have everything needed for the

Proof of Theorem 1.2: Assume first that Conjecture 1 is true, i.e., every graph has the Erdős-Hajnal property. Let T be a tournament. We want to show that T also has the Erdős-Hajnal property.

Choose an arbitrary ordering $<$ of the vertex set of T. Applying Theorem 3.4 to the ordered graph $(H(T), <)$ associated with T and $<$, we obtain that there exists a graph H' with the property that, for any ordering $<'$ of H', $(H(T), <)$ is an induced subgraph of $(H', <')$. By Conjecture 1, there exists an $\varepsilon(H') > 0$ such that every H'-free graph with n vertices has a homogeneous subset of size at least $n^{\varepsilon(H')}$.

Consider now a T-free tournament T' with n vertices and an ordering $<'$ of $V(T')$. Then the ordered graph $(H(T'), <')$ associated with them cannot contain an induced subgraph isomorphic to H' (because, no matter how it is ordered, this would yield a copy of T in T'). Thus, $H(T')$ must have a homogeneous set of size at least $n^{\varepsilon(H')}$. However, a homogeneous set in $(H(T'), <')$ corresponds to a transitive subtournament in T'.

The proof of the reverse statement is very similar, but the roles of graphs and tournaments have to be switched. Assume that Conjecture 2 is true, and let H be an arbitrary graph. To establish that H has the Erdős-Hajnal property, fix a linear order $<$ on $V(H)$, and denote the associated ordered tournament by $(T(H), <)$.

By Theorem 1.3, there exists a tournament T' with the property that, for any ordering $<'$ of T', $(T(H), <)$ is an induced subtournament of $(T', <')$. By Conjecture 2, there exists an $\varepsilon(T') > 0$ such that every T'-free tournament with n vertices has a transitive subtournament of size at least $n^{\varepsilon(T')}$.

Consider now an H-free graph H' with n vertices and an ordering $<'$ of $V(H')$. Then the ordered tournament $(T(H'), <')$ associated with them cannot contain a subtournament isomorphic to T' (because, no matter how it is ordered, this would yield a copy of H in H'). Thus, $T(H')$ must have a transitive subtournament of size at least $n^{\varepsilon(H')}$. However, by Lemma 4.1, any such subtournament has at least $n^{\varepsilon(H')/2}$ vertices such that, with respect to the ordering $<'$, either all edges connecting them are directed towards their larger endpoints, or all of them are directed towards their smaller endpoints. These vertices induce a complete or an empty subgraph of H', respectively. $\square$

References

[AS92] N. Alon and J. H. Spencer, *The Probabilistic Method*, Wiley, 1992.

[BLTV99] H. Buhrman, M. Li, J. Tromp and P. Vitanyi, Kolmogorov random graphs and the incompressibility method, SIAM J. Computing, to appear.

[EEH73] R. C. Entringer, P. Erdős and C. C. Harner: Some extremal properties concerning transitivity in graphs, *Periodica Mathematica Hungarica* **3** (1973), 275–279.

[E47] P. Erdős: Some remarks on the theory of graphs, *Bulletin of the American Mathematical Society* **53** (1947),292–294.

[EH89] P. Erdős and A. Hajnal: Ramsey-type theorems, *Discrete Mathematics* **25** (1989), 37–52.

[ES35] P. Erdős and G. Szekeres: A combinatorial problem in geometry, *Compositio Math.* **2** (1931), 463–470.

[G99] M. Ginn: Forbidden ordered subgraph vs. forbidden subgraph characterizations of graph classes, *Journal of Graph Theory* **30** (1999), 71-76.

[GHK73] D. L. Greenwell, R. L. Hemminger and J. Kleirlein: Forbidden subgraphs, in: *Proceedings of the Fourth Southeastern Conference on Combinatorics, Graph Theory and Computing (Florida Atlantic University, Boca Raton, FL, 1973), Utilitas Math.*, Winnipeg, Manitoba, 1973, 389–394.

[G97] A. Gyárfás: Reflections on a problem of Erdős and Hajnal, in: *The Mathematics of Paul Erdős* (R. L. Graham and J. Nešetřil, eds.), Algorithms and Combinatorics 14, Volume II, Springer-Verlag, Heidelberg, 1997, 93–98.

[L72] L. Lovász: Normal hypergraphs and the perfect graph conjecture, *Discrete Mathematics* **2** (1972), 253–267.

[L83] L. Lovász: Perfect Graphs, in: *Selected Topics in Graph Theory, Volume 2* (L. W. Beineke, R. J. Wilson, eds.), Academic Press, London-New York, 1983, 55–87.

[Se74] D. Seinsche: On a property of the class of n-colorable graphs, *Journal of Combinatorial Theory, Ser. B* **16** (1974), 191–193.

[S74] J. Spencer: Random regular tournaments, *Periodica Mathematica Hungarica* **5** (1974), 105–120.