Smallest limited vertex-to-vertex snakes of unit triangles

Heiko Harborth
Diskrete Mathematatik
Technische Universität Braunschweig
Pockelsstr. 14
D-38106 Braunschweig
Germany

László Szabó*
Computer and Automation Institute
Hungarian Academy of Sciences
Lágymányosi út 11
H-1111 Budapest
Hungary

Zoltán Ujváry-Menyhárt
Department of Geometry
Eötvös University
Rákóczi út 5
H-1088 Budapest
Hungary

February 26, 1999

Abstract

A sequence $\mathcal{T} = \langle T_1, T_2, \ldots, T_n \rangle$ of regular triangles of unit side lengths is called a vertex-to-vertex snake if $T_i \cap T_j$ is a common vertex

*The work was supported by Hungarian State Eötvös Fellowship and by Hungarian Scientific Research Fund No. F019449.

of T_i and T_j if $|i-j| = 1$ and is empty if $|i-j| > 1$. A vertex-to-vertex snake of unit triangles is called limited if it is not a proper subset of another vertex-to-vertex snake of unit triangles. We prove that the minimum number of unit triangles which form a limited vertex-to-vertex snake is seven.

1. Introduction

A sequence $\mathcal{C} = \langle C_1, C_2, \dots, C_n\rangle$ of pairwise non-overlapping congruent convex bodies in the plane where $C_i \cap C_j \neq \emptyset$ if and only if $|i-j| \leq 1$ is called a snake. If the snake $\mathcal{C}$ is not a proper subset of another snake of convex bodies congruent to the members of $\mathcal{C}$ then we say that the snake is limited. In other words, the snake $\mathcal{C}$ is limited if it cannot be continued at its ends with convex bodies congruent to the members of $\mathcal{C}$ without violating the snake property. It was proved in [1] that the minimum number of congruent circles which form a limited snake is 10. This result was extended in [3] for translates of arbitrary convex bodies. Namely, it was proved that the minimum number of translates of a convex body which form a limited snake is 11 if the body is a parallelogram and is 10 otherwise.

If we restrict ourselves to snakes of congruent regular polygons we can introduce other kinds of snakes as well. Let $\mathcal{T} = \langle T_1, T_2, \dots, T_n\rangle$ be a snake of regular k-gons of unit side lengths, $k \geq 3$. If $T_i \cap T_{i+1}$ is a common edge of T_i and T_{i+1}, $1 \leq i \leq n-1$, then the snake is called an edge-to-edge snake. To include a lot of nice configurations for $k \leq 5$ as well, in these cases it is allowed that consecutive members of $\mathcal{T}$ can have a common vertex, but other intersections are forbidden again. An edge-to-edge snake of regular k-gons of unit side lengths is called limited if it is not a proper subset of another edge-to-edge snake of regular k-gons of unit side lengths. Answering a problem of Harborth (see [4]), it was proved in [2], [5] and [6] that the smallest number of congruent squares which form a limited edge-to-edge snake is 19. It was also proved in [2] that the smallest number of congruent regular triangles and congruent regular hexagons which form a limited edge-to-edge snake is 20 and 13, respectively.

In this paper we will consider another variant of snakes. Let $\mathcal{T} = \langle T_1, T_2, \dots, T_n\rangle$ be a snake of regular k-gons of unit side lengths, $k \geq 3$, where $T_i \cap T_{i+1}$ is a common vertex of T_i and T_{i+1}, $1 \leq i \leq n-1$. Such snakes are called vertex-to-vertex snakes. A vertex-to-vertex snake of regular k-gons of unit side lengths is called limited if it is not a proper subset of another vertex-to-vertex snake of regular k-gons of unit side lengths. In this paper we prove the following theorem.

Theorem 1. *The minimum number of unit triangles which form a limited vertex-to-vertex snake is seven.*

2. Proof of Theorem 1

In this section vertex-to-vertex snakes of regular triangles of unit side lengths will be called snakes. We start with a simple observation.

Lemma 1. *Let $\mathcal{T}$ be a snake consisting of at least three triangles. Then, by reversing the order of the triangles in $\mathcal{T}$, if necessary, at least one of the lines containing the sides of T_1 does not intersect T_n and separates T_1 and T_n.*

Proof. Let l_1 and l_2 be the common support lines of T_1 and T_n which separate them. We can assume, by reversing the order of the triangles in $\mathcal{T}$, if necessary, that $l_1 \cap l_2$ is not a vertex of T_1. Let A, B, and C be the vertices of T_1 such that $A \in l_1$ and $B \in l_2$. If the line AB separates T_1 and T_n then we are done. On the other hand, if AB is not a separating line then both AC and BC are separating lines and at least one of these lines is clearly disjoint from T_n. □

First we prove that the number of triangles in a limited snake $\mathcal{T}$ is not six. By Lemma 1 there exists a line l which contains a side of T_1, does not intersect T_n, and separates T_1 and T_n. Suppose, for contradiction, that $\mathcal{T}$ consists of six triangles. Without loss of generality we may assume that the line l is horizontal and the triangle T_1 lies above l.

Throughout the proof we will use the following terminology. For any two points P and Q, we will say that P is above (below) Q if the closed half plane bounded by the horizontal line going through Q and being above (below) this boundary line contains P. Similarly, we will say that P is on the left of (on the right of) Q if the closed half plane bounded by the vertical line going through Q and being on the left of (on the right of) this boundary line contains P. Analogously, we can define these relations between sets as well. For instance, we say that a set p is above a set q if for any point $P \in p$ and for any point $Q \in q$ the point P is above Q. One more convention, the vertices of triangles are always listed in counterclockwise order.

Let A, B, C denote the vertices of T_1 so that $A, B \in l$. We will distinguish two cases, (1) $T_1 \cap T_2 \in l$, (2) $T_1 \cap T_2 \notin l$.

Case 1: $T_1 \cap T_2 \in l$

By symmetry, we may assume that $T_1 \cap T_2$ is B. Let T^I denote the unit triangle with vertices C, D, E where $\overline{CD}$ is vertical, D is above C, and E is on the left of C (see Figure 1). Since $\mathcal{T}$ is limited there exists a triangle T_i in $\mathcal{T}$, $i \geq 2$, which intersects T^I. Any unit triangle intersecting T^I must lie above l, therefore the triangles T_5 and T_6 cannot intersect T^I since $T_5 \cap T_6$ must be below l. Since $T_2 \cap T^I = \emptyset$ also holds we have only two possibilities, (1.1) $T_3 \cap T^I \neq \emptyset$, (1.2) $T_3 \cap T^I = \emptyset$ and $T_4 \cap T^I \neq \emptyset$.

Case 1.1. $T_3 \cap T^I \neq \emptyset$. — Now T_3 lies above l. Let B, F, G denote the vertices of T_2. Then T_2 lies above l and $T_2 \cap T_3$ is G. Since $\measuredangle GCD > \pi/3$ and the length of the segment $\overline{CG}$ is smaller than one, T_3 intersects the line CG at G only. This implies that the distance of the vertices of T_3 from l is not smaller than $\sqrt{3}/2 - \cos(\pi/3) > 0.366$ (see Figure 1).

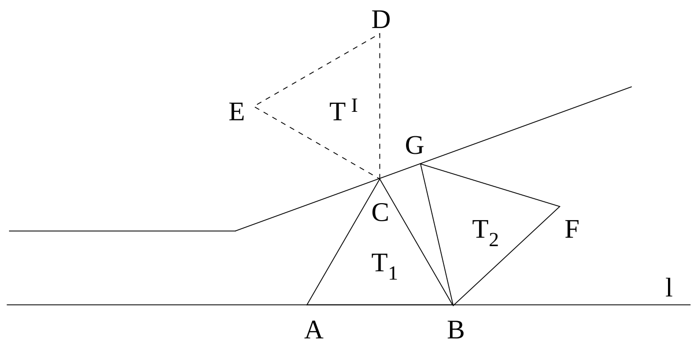

Figure 1.

We conclude, T_1, T_2, T_3 lie above l, and T_4 has a vertex above l whose distance from l is at least 0.366. The following lemma shows that this is impossible.

Lemma 2. *If T_1, T_2, T_3 lie above l and there is no vertex of T_4 below l whose distance from l is greater than $\sqrt{3}/2$ then $\mathcal{T}$ is not limited.*

Proof. Note that the distance of a lowest vertex of T_6 from l is greater than $\sqrt{3}/2$ since T_6 is strictly below l. Let l' be the horizontal line below l whose distance from l is $\sqrt{3}/2$. Let K, L, M denote the vertices of T_6 where K is a lowest vertex of T_6.

At first we assume that $T_5 \cap T_6$ is not a lowest vertex of T_6. By symmetry, we may assume that $T_5 \cap T_6$ is L. Let f be the open vertical half line with endpoint K and below K. The angle between f and the half line with endpoint K and through L is greater than $\pi/2$, hence $f \cap T_5 = \emptyset$. Let T^{II} and T^{III} denote the two unit triangles which have a common vertex at K, lie below K, and one side of each of them is horizontal (see Figure 2). If the lowest vertex of T_6 is not unique then we rotate these triangles around K by a suitable small angle towards f. The triangles T^{II} and T^{III} are disjoint from T_1, T_2, T_3, T_4 and they intersect T_6 at K only. Since $T_5 \cap f = \emptyset$ we can continue $\mathcal{T}$ with T^{II} or T^{III}, a contradiction.

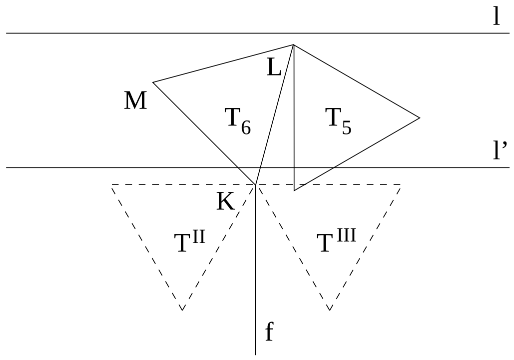

Figure 2.

Secondly, we assume that $T_5 \cap T_6$ is K. Then $T_4 \cap T_5$, say N, lies above l'. By symmetry, we may assume that KL separates $\overline{KM}$ and $\overline{KN}$. Then M is also a vertex of a unit triangle T^{IV} which has a horizontal side and lies below and on the left of M (see Figure 3).

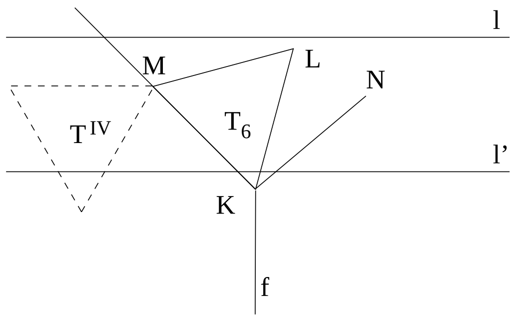

Figure 3.

The triangles T_1, T_2, T_3 are disjoint from T^{IV}. Also T_4 and T_5 are disjoint from T^{IV} since they have no points in common with f and with the open half line with endpoint K and through M. Therefore we can continue $\mathcal{T}$ with T^{IV}, a contradiction. □

Case 1.2. $T_3 \cap T^I = \emptyset$ and $T_4 \cap T^I \neq \emptyset$. — If both T_2 and T_3 lie above l then we obtain a contradiction by Lemma 2. So assume that T_2 is above l and T_3 has a vertex not above l. Then $T_2 \cap T_3$, say F, is the vertex of T_2 following B in the counterclockwise order. The vertex $T_3 \cap T_4$ must lie on the right of F since T_3 has a vertex below l. But the distance of F from CD is greater than one which implies that the T_2 must have a vertex lying not above l (see Figure 4).

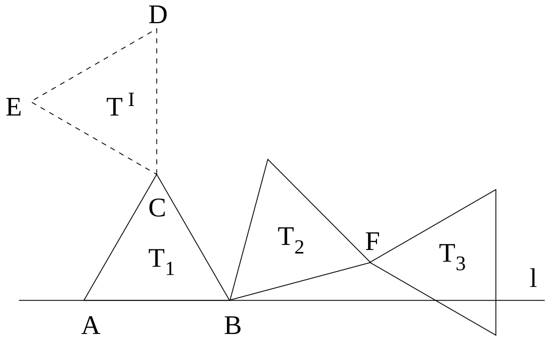

Figure 4.

Let l_1 and l_2 denote the open half lines on l with endpoint A and on the left of A and with endpoint B and on the right of B, respecytively. We will distinguish two cases, (1.2.1) $\mathcal{T}$ intersects first l_1, (1.2.2) $\mathcal{T}$ intersects first l_2.

Case 1.2.1. $\mathcal{T}$ intersects first l_1. — The triangle T_2 does not intersect l_2 or l_1, and T_4 is above l hence T_3 must intersect l_1. Let K be the point on the vertical line through A with distance one from A and above A. Then $T_3 \cap T_4$ is below and on the left of K (see Figure 5).

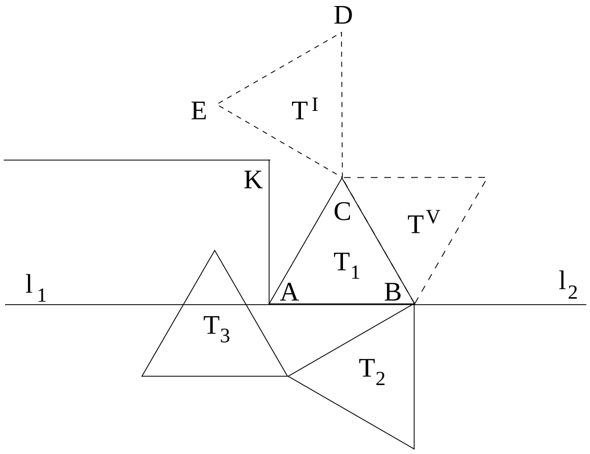

Figure 5.

Let T^V denote the image of T_1 under reflection with respect to the line BC. Since $\mathcal{T}$ is limited T_4 or T_5 must intersect T^V. Then the distance of $T_4 \cap T_5$ from l is greater than $1 - 2(1 - \sqrt{3}/2) = \sqrt{3} - 1$, that is, $T_5 \cap T_6$ is not a lowest vertex of T_6. The following lemma shows that this is impossible.

Lemma 3. *If $T_5 \cap T_6$ is not a lowest vertex of T_6, both T_2 and T_3 have vertices above l, and both T_4 and T_5 lie above a lowest vertex of T_6, then $\mathcal{T}$ is not limited.*

Proof. As in the proof of Lemma 2 let l' be the horizontal line below l whose distance from l is $\sqrt{3}/2$. Let K, L, M denote the vertices of T_6 where K is a lowest vertex of T_6. Again, let T^{II} and T^{III} denote the two unit triangles which have a common vertex at K, lie below K, and one side of each of them is horizontal (if the lowest vertex of T_6 is not unique then we rotate these triangles around K by a suitable small angle). Since $\mathcal{T}$ is limited, T_2 intersects exactly one of the triangles T^I and T^{II} while T_3 intersects the other triangle. If $T_2 \cap T_3$ lies below a lowest vertex of T_6 then T_6 lies in the triangle surrounded by T_2, T_3, and l, which is impossible. Similarly, if $T_2 \cap T_3$ lies above a lowest vertex of T_6 then T_6 must lie in the triangle surrounded by T_2, T_3, and the horizontal line going through a lowest vertex of T_6 which also is impossible. □

Case 1.2.2. $\mathcal{T}$ intersects first l_2. — As observed in Case 1.2 at least one vertex of T_2 lies not above l. Then T_2 or T_3 must intersect l_2. Let T^{VI} be the translate of T_1 whose rightmost vertex is A. Now T_4 or T_5 must intersect T^{VI} which implies that $T_5 \cap l_1 \neq \emptyset$ (see Figure 6).

If $T_5 \cap T_6$ is not a lowest vertex of T_6 then T_5 does not have a vertex below a lowest vertex of T_6 and thus $\mathcal{T}$ is not limited by Lemma 3. On the other hand, if $T_5 \cap T_6$ is a lowest vertex of T_6 then the distance of $T_4 \cap T_5$ from l is smaller than $1 - \sqrt{3}/2$. Now the distance of $T_3 \cap T_4$ from $T_4 \cap T_5$ and $T_2 \cap T_3$ can be one if at most one side of T_2 is below l and the angle between this side and l is smaller than $\pi/6$. Then $\mathcal{T}$ can be continued with the translate of T_1 whose topmost vertex coincides with a second lowest vertex of T_6 (see Figure 6).

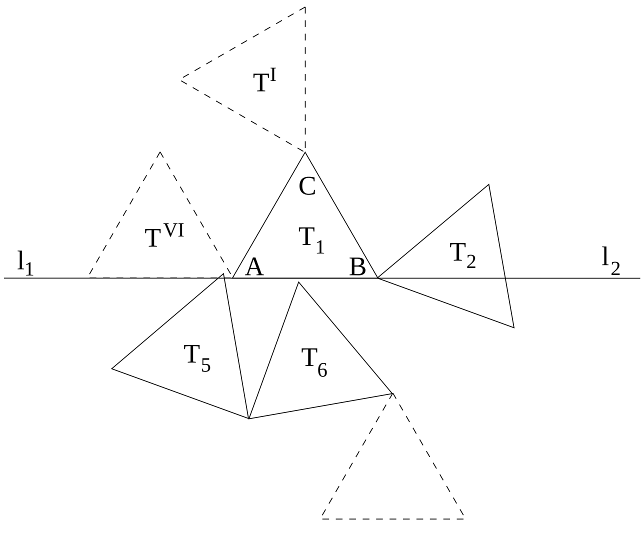

Figure 6.

Case 2: $T_1 \cap T_2 \not\in l$

This is the case when $T_1 \cap T_2$ is C. Let t be the open vertical half line with endpoint C and above C. If $T_2 \cap t \neq \emptyset$ then the distance of the vertices of T_2 different from C from l is at least $\sqrt{3}/2 + 1/2$. This implies that T_3 is above l and the distance of $T_3 \cap T_4$ from l is at least $\sqrt{3}/2 - 1/2 > 1 - \sqrt{3}/2$. Then by Lemma 2 the snake is not limited. By symmetry, we may assume that T_2 is on the left of t. As in Case 1.2.1 let T^V denote the image of T_1 under reflection with respect to the line BC. Since $\mathcal{T}$ is limited T_4 or T_5 must intersect T^V. We will distinguish two cases, (2.1) $T_4 \cap T^V \neq \emptyset$, (2.2) $T_4 \cap T^V = \emptyset$ and $T_5 \cap T^V \neq \emptyset$.

Case 2.1. $T_4 \cap T^V \neq \emptyset$. — If $T_3 \cap T_4$ is above l then $T_3 \cap T_4$ must be on the right of t hence T_3 must intersect t. Now the distance of $T_3 \cap T_4$ from l is at least $\sqrt{3}/2$ which implies by Lemma 2 that $\mathcal{T}$ is not limited. Therefore $T_3 \cap T_4$ is below l.

Let e and h be the half lines below l with endpoint A and perpendicular to AC and with endpoint B and parallel to AC, respectively. Furthermore, let $\mathcal{R}$ be the convex region below $e \cap h$ surrounded by e, h, and the unit circle around A. Since T_3 and T_4 cannot intersect e and h, respectively, $T_3 \cap T_4$ must lie in the interior of $\mathcal{R}$. Let D, E, F denote the vertices of T_4 so that $T_3 \cap T_4$ is E (see Figure 7). Then E is the lowest vertex of T_4 and D lies in T^V.

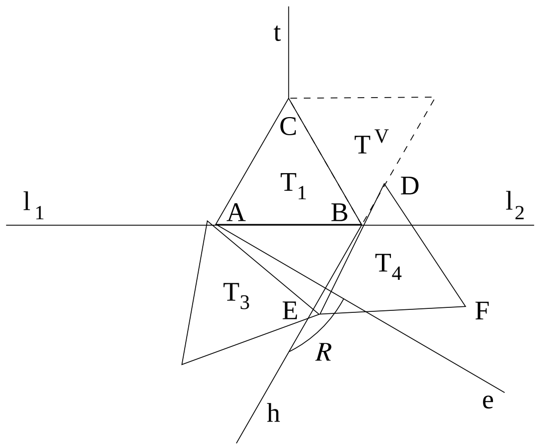

Figure 7.

If T_6 does not intersect ED then we can apply Lemma 2 with ED instead of l to conclude that $\mathcal{T}$ is not limited. Indeed, we did not use in the proof of Lemma 2 that l contains a side of T_1.

Thus we can assume that T_6 intersects ED. Now $T_4 \cap T_5$ must be F. Let F, G, H denote the vertices of T_5. If $T_5 \cap T_6$ is H then, denoting by H, I,

J the vertices of T_6, we can continue $\mathcal{T}$ with the translate T^{VII} of T_1 whose topmost vertex coincides with J since J is below both T_5 and T_6. On the other hand, if $T_5 \cap T_6$ is G then, denoting by G, I, J the vertices of T_6, we can also continue $\mathcal{T}$ with T^{VII}. This is clear if J is the lowest vertex of T_6, and if J is not a lowest vertex of T_6 then G must be below H implying that T^{VII} intersects T_6 at J only (see Figure 8).

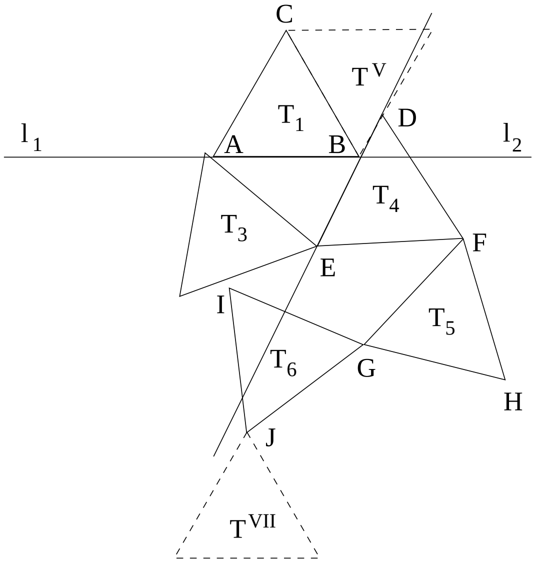

Figure 8.

Case 2.2. $T_4 \cap T^V = \emptyset$ and $T_5 \cap T^V \neq \emptyset$. — If $T_3 \cap l = \emptyset$ and $T_4 \cap l = \emptyset$ then by Lemma 2 the snake is not limited. If $T_3 \cap l = \emptyset$ and $T_4 \cap l \neq \emptyset$ then T_4 intersects l_1 since $T_4 \cap T^V = \emptyset$. The vertex $T_4 \cap T_5$ is in the interior of the union of $\mathcal{R}$ defined in Case 2.1 and its reflection with respect to e. The vertex $T_4 \cap T_5$ is the lowest vertex of T_5 and its distance from l is smaller than $\sqrt{3}/2$. This implies that $T_5 \cap T_6$ is not a lowest vertex of T_6. Furthermore, at most one vertex of T_4 is below a lowest vertex of T_6. The following slight modification of Lemma 3 shows that this is impossible.

Lemma 4. *If $T_5 \cap T_6$ is not a lowest vertex of T_6, both T_3 and T_4 have vertices above l, and both T_2 and T_5 lie above a lowest vertex of T_6, then $\mathcal{T}$ is not limited.*

Thus T_3 intersects l_1. If T_5 has two vertices above l then the distance of $T_5 \cap T_6$ from l is smaller than $\sqrt{3}/2$ hence $T_5 \cap T_6$ is not a lowest vertex of T_6. Since T_3 and T_4 both have vertices above l at most one of their vertices is below a lowest vertex of T_6. Then $\mathcal{T}$ is not limited by Lemma 4.

Thus only one vertex of T_5 is above l and this vertex is in T^V. Let D, E, F denote the vertices of T_5 so that $D \in T^V$. Now F is above E and the distance of F from l is smaller than $\sqrt{3}/2$. First we assume that $T_4 \cap T_5$ is not F. Then $T_3 \cap T_4$ must be in the interior of $\mathcal{R}$ defined in Case 2.1. Hence the distance of the points of the region surrounded by l and the polygonal line with vertices $T_2 \cap T_3$, $T_3 \cap T_4$, F and D from l is smaller than $\sqrt{3}/2$. But this region must contain T_6 which is impossible. Therefore $T_4 \cap T_5$ and $T_5 \cap T_6$ must be E and F, respectively.

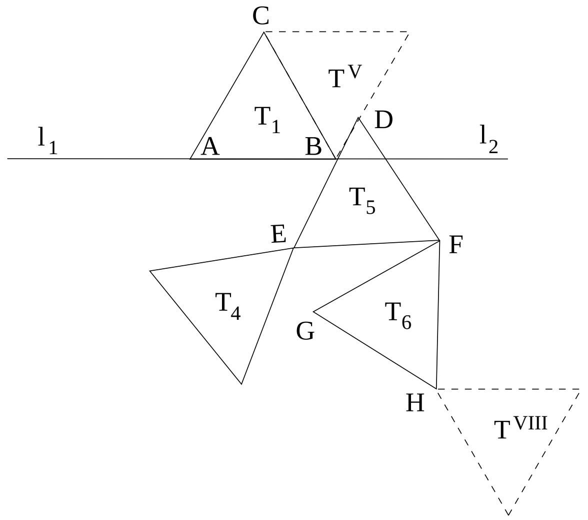

Figure 9.

Let F, G, H denote the vertices of T_6. Let T^{VIII} be the translate of T^V with a vertex at H and on the right of H (see Figure 9). Since T_6 intersects T^{VIII} at H only, and T_3, T_4, T_5 do not intersect T^{VIII} we can continue $\mathcal{T}$ with T^{VIII}. This finishes the proof that six unit triangles cannot form a limited vertex-to-vertex snake. Following the line of the above proof one can easily show that less than six unit triangles also cannot form a limited vertex-to-vertex snake.

To complete the proof we describe a limited vertex-to-vertex snake consisting of seven unit triangles. Let A_i, B_i, C_i denote the vertices of the unit triangle T_i, $1 \leq i \leq 7$, where $A_{i+1} = C_i$, $1 \leq i \leq 6$. Let $\measuredangle C_2A_2A_1 = \measuredangle C_7C_6A_6 = \pi/180$, $\measuredangle C_3C_2A_2 = \measuredangle C_6A_6A_5 = \pi/2$ and $\measuredangle C_4A_4A_3 = \measuredangle C_5A_5A_4 = 2\pi/3 - \pi/180$ (see Figure 10). Then B_1C_1 and B_7A_7 are parallel and their distance from each other is $1+2\cos(\pi/3+\pi/180) - 2\sin(\pi/3+\pi/180) < 0.3$. Also, the distance between the parallel lines B_1B_7 and A_4A_5 is $\sin(\pi/3 + \pi/180) + \cos(\pi/3 + \pi/180) - 1 < 1/2$. These observations imply that the snake described above is limited.

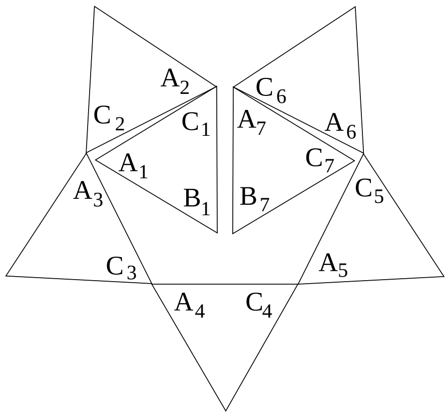

Figure 10.

This finishes the proof of Theorem 1.

3. Concluding remarks

In this paper we have proved that the minimum number of unit triangles which form a limited vertex-to-vertex snake is seven. Next one can ask for the corresponding minimum number of unit squares instead of unit triangles. It is not hard to find a limited vertex-to-vertex snake with 8 unit squares, and it is very likely that less than 8 squares do not form a limited vertex-to-vertex snake. Unfortunately, the problem turns out to be much harder. More difficulties can be expected if one realizes that there are (at least) two totally different constructions using 8 squares (see Figure 11).

Finally, we would like to mention that if k is large enough then one can construct limited vertex-to-vertex snakes consisting of 10 regular k-gons with unit side lengths. The construction is similar to the construction given in [1] for circles. Moreover, it is very likely that less than 10 regular k-gons with unit side lengths do not form a limited vertex-to-vertex snake if k is large enough. Thus the following conjecture might be posed.

Conjecture 1. *The minimum number of regular k-gons with unit side lengths which form a limited vertex-to-vertex snake is 10 if k is large enough.*

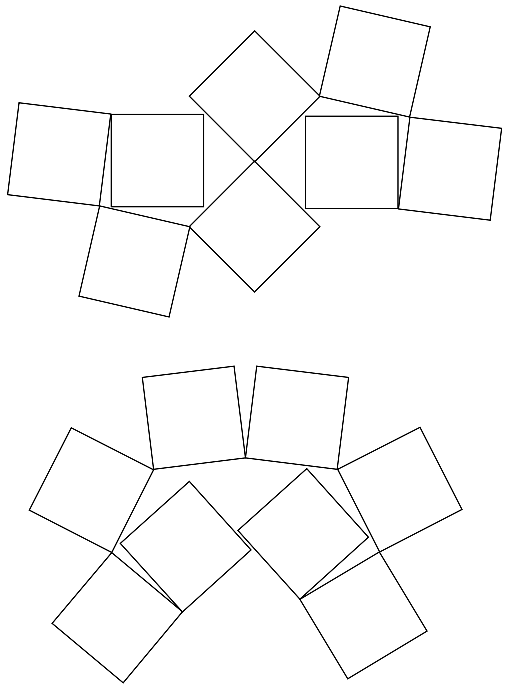

Figure 11.

References

[1] Bisztriczky, T., Böröczky Jr, K., Harborth, H., Piepmeyer, L.: On the smallest limited snake of unit disks, *Geometriae Dedicata* **40** (1991), 319-324.

[2] Bisztriczky, T., Harborth, H.: On the smallest limited edge-to-edge snakes of congruent regular triangles, squares, and hexagons, Manuscript.

[3] Böröczky Jr, K., Soltan, V.: Smallest maximal snakes of translates of convex domains, *Geometriae Dedicata* **54** (1995), 31-44.

[4] Harborth, H.: Problem 45: Kleinste endliche Schlange, *Math. Semesterber.* **36** (1989), 269-270.

[5] Heidelberg, R., Stege, L., Weiß, H.: Lösung zu Problem 45, *Math. Semesterber.* **38** (1991), 137-138.

[6] Hering, F.: Beweis einer Vermutung von Heiko Harborth über Polyominos aus Quadraten, *Math. Semesterber.* **38** (1991), 223-237.