Recent results on illumination problems

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1. Introduction

In this paper we survey combinatorial type theorems about illumination or visibility of certain geometric objects, e.g. polygons, polytopes, convex bodies. A typical question of interest here is the following. How many light sources are needed to illuminate (in an appropriate sense) the boundary of one or more convex sets in the d-dimensional Euclidean space E^d. As it turned out, illumination problems are closely related to some deep theorems in graph theory, in the theory of polytopes and in the theory of packing and covering. Also, some of the problems have been found related to rapidly developing areas of computer sciences. The selection of topics in this survey is not intended to form an account of all research accomplishments in the area, but rather to illustrate some of the recent trends and new techniques. The contents of this survey have also been affected by the author's personal taste, thus certain results, even some very interesting ones, have not been included.

The paper is organized as follows: 2. Krasnosselski's theorem, 3. Art gallery problems, 4. Three dimensional art galleries, 5. Illumination of families of convex sets, 6. Hadwiger's problem.

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2. Krasnosselski's theorem

Let $S \subseteq E^d$ be a compact set that contains at least $d+1$ points. A well-known theorem of Krasnosselski [34] states that if for each $d+1$ points of S there is a point of S from which all $d+1$ points can be seen, then there is a point of S from which all points of S can be seen. Here a point $p \in S$ sees a point $q \in S$ if the closed line segment $[p,q] \subseteq S$. Colloquially, imagine an art gallery consisting of several connected rooms in which the walls are completely covered with pictures. The theorem implies that if for every three paintings in the gallery there exists a point from which all three paintings can be seen, then there exists a point from which all the paintings in the gallery can be seen. The combinatorial flavour of this result is very appealing and it has inspired some interesting questions.

If $p, q \in S$ and p sees q, we also say that p illuminates q. We now extend this concept of illumination. Let $P, Q \subseteq S$. Then P illuminates Q if for each point of $q \in Q$ there is a point $p \in P$ that illuminates q. First we restrict ourselves to planar problems. It is natural to ask if any k points of a simply connected compact set S are illuminated by some line segment of S, does S contain a line segment illuminator? Connelly has shown that the answer is no, regardless of how large we choose k (see [1]). Thus, to obtain positive results we need extra conditions on the line segments. For example, Bezdek, Bezdek and Bisztriczky proved in [1] that if any two points of a simply connected compact set S are illuminated by some line segment in S parallel to a line l, then S contains a line segment illuminator parallel to l. They also showed that if any three points of a simply connected compact set S with smooth boundary are illuminated by some translate in S of a line segment T, then S contains an illuminator which is a translate of T.

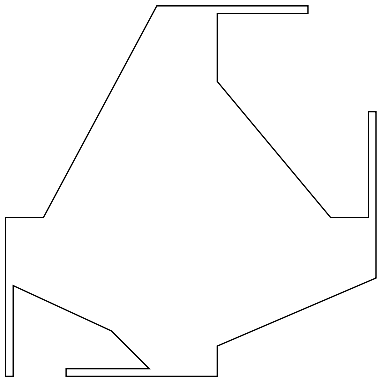

Figure 1.

A simple construction shows the existence of a simply connected compact set S such that any three points of S are illuminated by some convex subset of S but S does not contain a convex illuminator (see Figure 1). This example raises the question of the existence of a positive integer $k > 3$ with the property that if any k points of a simply connected compact set S are illuminated by a convex subsets of S, then S contains a convex illuminator. In connection with this problem one might ask if a simply connected compact set S, any three points of which are illuminated by a line segment in S, contains a convex illuminator. The only known result in this direction is the following theorem of Bezdek, Bezdek and Bisztriczky [1]. If any three points of an o-symmetric simple polygon P are illuminated by a line segment in P through the point o, then P contains an o-symmetric convex illuminator which is either a line segment or a parallelogram.

The next problem can also be described colloquially. What simple condition will enable a group of swimmers to swim in straight path to a common point on the shoreline of a convex island? If we assume that the sea is flat, the answer is simple although unexpected. It is true that if any five swimmers can swim linearly to some common point of the shoreline, then all can swim linearly to some common point.

To make this precise, we introduce the concept of external illumination. A point $p \in E^d$ exterior to a compact convex set $S \subseteq E^d$ is said to be illuminated by a boundary point $q \in \mathrm{bd}S$ if and only if $[p, q] \cap S = \{q\}$. Now, the above theorem, proved by Valentine [55], can be formulated as follows. Let S be a compact convex set in the plane and let P be a collection of points lying outside S. If any five or fewer points in P can be illuminated by some point of $\mathrm{bd}S$, then there exists at least one point $q \in \mathrm{bd}S$ which illuminates all the points of P. We note that the number five in the above statement is the best possible.

Valentine also proved in [55] that if any $d + 1$ or fewer points of a set P exterior to a compact strictly convex set $S \subseteq E^d$ with smooth boundary can be illuminated by some point of $\mathrm{bd}S$, then there exists a point of $\mathrm{bd}S$ which illuminates all the points of P. This theorem raises the question of the existence of a positive integer $h(d)$ with the property that if any $h(d)$ or fewer points of a set P exterior to a compact convex set $S \subseteq E^d$ can be illuminated by some point of $\mathrm{bd}S$, then there exists a point of $\mathrm{bd}S$ which illuminates all the points of P. In connection with this problem, for every $n \geq 3$, Buchman and Valentine [13] constructed a convex polyhedron with $4n$ vertices in E^3 with the property that each collection of $4n - 1$ vertices of this polyhedron can illuminate some exterior point, but there is no exterior point which is illuminated by all $4n$ vertices. We note that similar examples exist in higher dimensions as well. These results easily imply that the answer to Valentine's above question is in the negative.

3. Art gallery problems

Art gallery problems ask the following question. How many guards are always sufficient and sometimes necessary to see every point in an art gallery. The gallery is viewed as a simple polygon and a guard is a point in the polygon. If this point coincides with a vertex of the polygon, then the guard is called a vertex guard. If there is no restriction on the location of the point, then the guard is called a point guard. We say that a guard p sees a point q of the polygon if and only if the closed line segment $[p, q]$ is contained in the polygon. The original art gallery problem, posed in 1973 by Klee, asks for the number of guards required to protect any art gallery with n walls and no interior obstruction. The problem was soon solved by Chvátal [15], who showed that $\lfloor n/3 \rfloor$ vertex guards are always sufficient and sometimes necessary.

A different and strikingly simple sufficiency proof of the same result was given by Fisk [22]. Fisk's proof starts with an arbitrary triangulation of the polygon. The second step is to three-colour the graph of the triangulated polygon: assign to each vertex one of three different colours so that no two adjacent vertices have the same colour. In the third step the guards are placed at the vertices of the least-used colour. By the pigeonhole principle, at most $\lfloor n/3 \rfloor$ vertices can be coloured by this colour, so no more than $\lfloor n/3 \rfloor$ guards are used. Finally, since all three vertices of a triangle have different colours, every triangle has a guard at one of its vertices.

If the art gallery is allowed to have obstructions (e.g. pillars) in its interior, the corresponding floor plan is a simple polygon with other disjoint simple polygons called holes inside. In such a polygon $\lfloor n/3 \rfloor$ guards are no longer sufficient. Shermer established in 1982 that $\lfloor (n+h)/3 \rfloor$ guards are necessary for certain polygons with n vertices and h holes (see [41]). It took almost ten years before Bjorling-Sachs and Souvaine [6] and Hoffmann, Kaufmann and Kriegel [32] independently proved that $\lfloor (n+h)/3 \rfloor$ point guards are also sufficient. The problem for vertex guards remains open.

There are two interesting variants of the original art gallery problem, the fortress problem and the prison yard problem. The first asks for the number of guards needed to see the exterior of a polygon, and the second asks for the number of guards needed to see both the exterior and the interior. O'Rourke and Wood proved that $\lceil n/2 \rceil$ vertex guards are always sufficient and sometimes necessary to see the exterior of a polygon with n vertices. Furthermore, Aggarwal and O'Rourke showed that $\lceil n/3 \rceil$ guards are always sufficient and sometimes necessary to see the exterior of a polygon with $n > 3$ vertices if the guards may be located anywhere in the plane exterior to the polygon or on the boundary of the polygon. Both results with their simple proofs can be found in [41]. However, the prison yard problem turned out to be much more complicated. It is easy to see that a convex polygon with n vertices requires at most $\lceil n/2 \rceil$ vertex guards and indeed needs that many. Confirming a conjecture of O'Rourke (see [41]), Füredi and Kleitman proved in [23] that $\lceil n/2 \rceil$ vertex guards always suffice to see both the exterior and the interior of a simple polygon with n vertices and in the non-convex case $\lfloor n/2 \rfloor$ guards suffice. The proof is based on the following very complicated colouring lemma which is also interesting in its own right. Let $\mathcal{R}$ be a planar triangulated figure with n vertices, each of whose external edges is labeled with at most two labels chosen from the set $\{\alpha, \beta\}$. Edges which are external on both sides can have both labels, while the other external edges can have at most one label. Let q be any vertex on the boundary of $\mathcal{R}$. Then there are subsets A and B of the vertex sets of $\mathcal{R}$ such that (1) each triangle of $\mathcal{R}$ intersects both A and B, q lies in B, (2) every external edge of $\mathcal{R}$ intersects the set corresponding to its label(s), (3) the sum of the cardinalities of A and B is at most n.

Another interesting variant of the original art gallery problem is the so-called α-guard or α-floodlight problem. An α-guard sees only in a cone of angle α. First we note that, for every $\alpha < \pi$, Estivill-Castro, O'Rourke, Urrutia and Xu [20] constructed simple polygons with n vertices with the property that no collection of n guards with vision cone of angle α, placing one guard at each vertex, can see the interior of the polygon. On the other hand, Csizmadia and Tóth showed in [19] that $\lceil 2(n-3)/5 \rceil$ guards with vision cone of angle π are always sufficient for a polygon with $n > 3$ vertices. Perhaps the correct answer is again $\lfloor n/3 \rfloor$.

Similar questions can be considered for restricted classes of polygons (e.g. orthogonal polygons, monotone polygons) and for another types of guards (e.g. mobile guards). For further information we refer the reader to the monograph [41] and the survey [45].

We close this section with a slightly different problem. Let X be a simply connected compact set in the plane of Lebesgue measure $\lambda(X) = 1$. For a point $x \in X$ let $V(x)$ denote the set of all points $y \in X$ seen by x. Kalai and Matoušek proved in [33] (see also [56]) that if there exists a positive real number ε such that $\lambda(V(x)) \geq \varepsilon$ for all $x \in X$, then X can be guarded by at most $C\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$ points where C is a universal constant. Certain ideas used in the proof might turn out to be useful tools in several related problems, so we briefly describe them here.

First we recall some definitions. Let $\mathcal{S}$ be a set system on a set X. We say that a subset $A \subseteq X$ is shattered by $\mathcal{S}$ if every possible subset of A can be obtained as the intersection of some $S \in \mathcal{S}$ with A. The VC-dimension of $\mathcal{S}$ is the supremum of the sizes of all finite shattered subsets of X. Let μ

be a probability measure on X such that all sets of $\mathcal{S}$ are measurable. For a fixed positive real number ε, a set $N \subseteq X$ is called an ε-net for $\mathcal{S}$ with respect to μ if it intersects each $S \in \mathcal{S}$ with $\mu(S) \geq \varepsilon$. It is well known that if the VC-dimension of $\mathcal{S}$ is at most r, then for any $0 < \varepsilon < 1$ there exists an ε-net for $\mathcal{S}$ (with respect to μ) of size at most $C(r)\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$ where the constant $C(r)$ depends on r only (see [31]).

Returning to the proof of the theorem, consider a simply connected compact set X in the plane of Lebesgue measure $\lambda(X) = 1$ and suppose that $\lambda(V(x)) \geq \varepsilon$ for all $x \in X$. It can be shown that the VC-dimension of the set system $\mathcal{V}(X) = \{V(X) : x \in X\}$ is bounded by a constant. Now, the result mentioned above implies the existence of an ε-net for $\mathcal{V}(X)$ (with respect to the Lebesgue measure) of size at most $C\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$. This ε-net intersects each $V(x)$ and thus guards X.

4. Three dimensional art galleries

Very little is known about art galleries in three dimensions. To understand the difficulties first we investigate the problem of partitioning a 3-dimensional polyhedron into tetrahedra (in a partition only vertices from the original polyhedron may be used as vertices of the tetrahedra and overlaps are allowed only in common vertices, edges and polygonal faces). Partitioning a polygon into triangles has been well studied and many efficient methods are known for producing triangulations, with the recent linear-time algorithm of Chazelle [14] being asymptotically optimal. However, the 3-dimensional triangulation problem is much more difficult than the 2-dimensional triangulation problem. One difference between the problems lies in the size of the resulting partitions. Every triangulation of an n-sided simple polygon produces $n-2$ triangles, but the number of tetrahedra in a triangulation of a given polyhedron is not unique. For example, a bipyramid with triangular base may be partitioned into either two or three tetrahedra.

Even more significant is the difference that any polygon may be triangulated, whereas there exist 3-dimensional simple polyhedra which cannot be triangulated. Probably the simplest example is due to Schönhardt [43]. This six-vertex polyhedron P can be constructed as follows. Start with a triangular prism and twist the (triangular) top face by a small amount. Now, the three side faces cannot remain planar, so allow them to break along the appropriate diagonals to become two triangular faces (see Figure 2). The polyhedron P cannot be triangulated because no tetrahedron formed by four of its vertices is contained in P. Indeed, if a tetrahedron was contained in P, then each pair of vertices of the tetrahedron would be able to see each other within P. Observe that the pairs of vertices of P which can see each other are exactly those which form the edges of P. Since the edge graph of P contains no complete subgraph on four vertices, P contains no tetrahedron. We note that the problem of deciding whether a given polyhedron can be triangulated is NP-complete even for the class of starshaped polyhedra (see [42]).

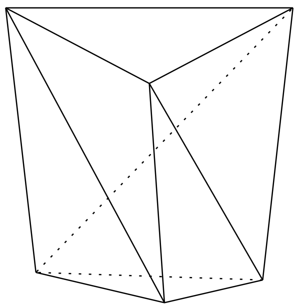

Figure 2.

In spite of the existence of non-triangulated polyhedra it seems almost obvious that guards placed at every vertex of a polyhedron see the entire interior of the polyhedron. However, this is not the case. Seidel constructed polyhedra of n vertices which require about $n^{3/2}$ guards to see the interior (see [41]).

The known (positive) art gallery theorems in three dimensions investigate the number of vertex guards needed to see the exterior of a convex polyhedron. The equivalent problem in two dimensions is trivial, $\lceil n/2 \rceil$ vertices are always sufficient and necessary to see the exterior of a convex polygon of n vertices. But in three dimensions the situation is not as straightforward. Grünbaum and O'Rourke proved in [41] that $\lfloor (2f-4)/3 \rfloor$ vertex guards are always sufficient and sometimes necessary to see the exterior of a convex polyhedron of $f \geq 10$ faces. On the other hand, Szabó and Talata proved in [54] that $\lfloor n/2 \rfloor$ (or $\lfloor n/2 \rfloor - 1$, if $n \equiv 2 \bmod 4$) vertex guards are always sufficient and sometimes necessary to see the exterior of a convex polyhedron of n vertices (the special case $n \equiv 3 \bmod 4$ was proved independently by Bose, Shermer, Toussaint and Zhu [12]). For comparison, if $f > \lfloor 3n/4 \rfloor + 2$, then $\lfloor n/2 \rfloor < \lfloor (2f-4)/3 \rfloor$.

5. Illumination of families of convex sets

Let $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$ be a family of n disjoint convex bodies (compact convex sets with non empty interior) in E^d. We say that a set V of exterior light sources illuminates $\mathcal{C}$, if for every boundary point u of any member of

$\mathcal{C}$ there is a point v in V such that u is visible from v, i.e. the open line segment joining u and v is disjoint from the members of $\mathcal{C}$. In this case V is called an illumination system for $\mathcal{C}$.

Fejes Tóth proved in [21] that $n \geq 4$ disjoint convex bodies in the plane can always be illuminated by $4n-7$ points. More precisely, he proved that $4n-7$ points are always sufficient and sometimes necessary to illuminate $n \geq 4$ disjoint bounded open convex sets in the plane. Recently, Czyzowicz, Gaujal, Rivera-Campo, Urrutia and Zaks considered the higher dimensional problem and they showed in [16] that for every convex body $S \subseteq E^d$ there exists a constant c (depending on S) such that any collection of n disjoint congruent copies of S can be illuminated by cn points.

Concerning families of balls, using a generalization of the so-called Upper Bound Theorem (see [53]), Linhart proved in [38] that any collection of n non-overlapping balls in E^d where the centres do not lie on a hyperplane can be illuminated by $\binom{n-\lfloor d/2 \rfloor}{n-d} + \binom{n-\lfloor (d+1)/2 \rfloor}{n-d}$ points.

From now on we restrict ourselves to planar problems. Czyzowicz, Rivera-Campo and Urrutia proved in [17] that $\lfloor (4n+4)/3 \rfloor$ points are always sufficient to illuminate n disjoint isothetic rectangles (rectangles with sides parallel to the coordinate axes). If, in addition, the rectangles are required to have equal horizontal width, then $n+1$ points suffice and $n-1$ are sometimes necessary. For n triangles, they showed that $\lfloor (4n+4)/3 \rfloor$ points are always sufficient and for any collection of n homothetic triangles $n+1$ points suffice and $n-1$ are occasionally needed. The best lower bounds in the general cases coincide with the ones given for the restricted cases.

Many people have studied illumination problems on arrangements of disjoint line segments. According to the definition of illumination of convex bodies, line segments need only be illuminated from one side. For this type of illumination, Czyzowicz, Rivera-Campo, Urrutia and Zaks proved in [18] that any family of $n \geq 11$ line segments can always be illuminated by $\lceil 2n/3 \rceil$ points. If the line segments are either vertical or horizontal, then $\lceil (n+1)/2 \rceil$ points suffice. Furthermore, Zaks [57] gave examples of families of $9k+5$ line segments which requires $4k+2$ points and of orthogonal families of $5k+3$ line segments which requires $2k+2$ points, for every $k \geq 1$.

We close this section with the dual of the original problem of Fejes Tóth. An illumination system V for a family $\mathcal{C}$ of disjoint convex bodies is called primitive if no proper subset of V illuminates $\mathcal{C}$. Let $p_{\max}(\mathcal{C})$ denote the maximum number of points forming a primitive illumination system for $\mathcal{C}$ (set $p_{\max}(\mathcal{C}) = \infty$ if the maximum does not exist). Furthermore, let $p_{\max}(n)$ denote the minimum of $p_{\max}(\mathcal{C})$ taken over all families $\mathcal{C}$ consisting of n disjoint convex bodies. These quantities were first introduced by Soltan [50] and investigated by Soltan, Szabó and Vásárhelyi [52]. They proved that if $p_{\max}(\{C_1, C_2, \ldots, C_n\})$ is finite, then all the bodies $C_1, C_2, \ldots, C_n$ are convex polygons. They also observed that, without additional assumptions the converse of this statement is not true. Indeed, one can find a family $\{T_1, T_2, T_3\}$ of three disjoint triangles such that $p_{\max}(\{T_1, T_2, T_3\}) = \infty$ (see Figure 3).

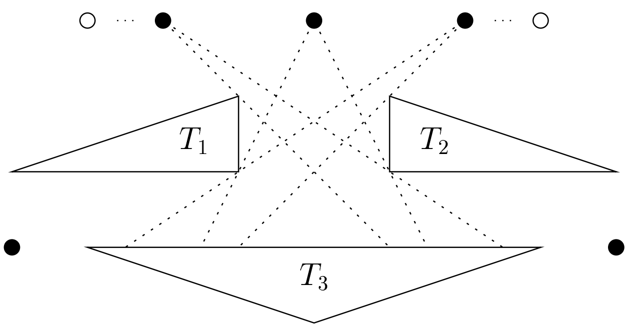

Figure 3.

To give a sufficient condition for the finiteness of $p_{\max}(\mathcal{C})$ the authors of [52] introduced the notion of critical triple (which is a rather complicated configuration condition for three disjoint convex polygons) and they showed that if a family $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ of n disjoint convex polygons contains no critical triple, then $p_{\max}(\mathcal{C})$ is finite. Unfortunately, (if the number of bodies is greater than three) general sufficient and necessary condition for the finiteness of $p_{\max}(\mathcal{C})$ is not known.

Turning to the problem of determining or estimating the quantity $p_{\max}(n)$, it is also proved in [52] that $p_{\max}(2) = 4$, $p_{\max}(3) \leq 6$ and $p_{\max}(n) \leq 3n - 2$ for $n \geq 4$. We note that the correct value of $p_{\max}(n)$ is not known for $n \geq 3$.

6. Hadwiger's problem

One of the most challenging open problems in discrete geometry is the following conjecture published by Hadwiger [28], [30], Levi [37] and Gohberg and Markus [24]. Any convex body $C \subseteq E^d$ can be covered by at most 2^d smaller homothetic copies of itself. Let $H(C)$ denote the least integer h such that C can be covered by h smaller homothetic copies of itself. It is easy to see that $H(C) = 2^d$ for parallelepipeds, and it is also conjectured that in all other cases $H(C) < 2^d$. For $d = 2$ this was shown by many authors (see [4], [7], [24], [37]).

Boltyanski [7] has suggested a surprising illumination approach to Hadwiger's problem. Let $C \subseteq E^d$ be a convex body and let q be a boundary point of C. We say that q is illuminated by a direction v if and only if the open ray emanating from q and having direction vector v intersects the interior of C. Let $I'(C)$ denote the minimum number of directions needed to illuminate every boundary point of C. Boltyanski made the simple but rather striking discovery that $H(C) = I'(C)$ for all convex bodies C. Furthermore, Soltan [46] observed that $H(C)$ also coincides with the minimum number $I(C)$ of exterior points needed to illuminate every boundary point of C. Here a point p in the exterior of C illuminates a boundary point q if the open ray emanating from p and having direction vector $\overrightarrow{pq}$ intersects the interior of C but there are no points of C on the open line segment $]p, q[$.

In spite of the fact that the problem has attracted a lot of attention, very little is known about it even in three dimensions. Lassak confirmed the conjecture for three-dimensional convex bodies of constant width (see [36]), and for three dimensional centrally symmetric convex bodies (see [35]). For centrally symmetric polyhedra in E^3, another proof was found by Soltan and Soltan [48]. Bezdek [2] generalized this result to convex polyhedra in E^3 with any affine symmetry.

Boltyanski [7] observed that if $C \subseteq E^d$ has at most d corners then $I(C) = d+1$. Schramm [44] confirmed the conjecture for convex bodies of constant width (in $d \geq 16$ dimensions). For zonotopes other than parallelepipeds, Martini [40] established the stronger bound $I(C) \leq 3 \cdot 2^{d-2}$. Using an integral representation of zonoids, Boltyanski and Soltan [11] extended this result for zonoids. Very recently, the same upper bound was verified for belt bodies as well (see [9]). It should be noticed that belt bodies form a dense set in the space of convex bodies. For centrally symmetric convex bodies, the estimate $I(C) \leq 2^d(d \ln d + d \ln \ln d + 5d)$ was obtained by Rogers (see [26]). We note that Roger's argument can be easily modified to give the estimate $I(C) \leq \binom{2d}{d}(d \ln d + d \ln \ln d + 5d)$ for general convex bodies.

Recently, Zong [58] has suggested a completely different approach to Hadwiger's problem for centrally symmetric convex bodies. Let $Z(C)$ denote the minimum number of non-overlapping translates of a given convex body $C \subseteq E^d$ so that they can touch C and prevent any other translate of C from touching C without overlapping each other. As a first step, Zong proved that $H(C) \leq Z(C)$ for all centrally symmetric convex bodies C. We present here his elegant argument.

Let $C \subseteq E^d$ an o-symmetric convex body and let $C + x_1, \ldots, C + x_n$ be non-overlapping translates of C so that they touch C and prevent any other translate of C from touching C without overlapping each other. Replace $C, C + x_1, \ldots, C + x_n$ by the bodies $2C, 2C + x_1, \ldots, 2C + x_n$. It is easy to see that

$$\operatorname{bd}(2C) \subseteq \bigcup_{i=1}^{n} (\operatorname{int}(2C) + x_i)$$

and hence

$$2C \setminus \{o\} \subseteq \bigcup_{i=1}^{n} (\operatorname{int}(2C) + x_i).$$

Now, using the simple observation $C \cap (\operatorname{int}(C) + x) \subseteq C \cap (\operatorname{int}(C) + \frac{1}{2}x)$, for all $x \in E^d$, we obtain

$$2C \subseteq \bigcup_{i=1}^{n} (\operatorname{int}(2C) + \tfrac{1}{2}x_i),$$

from which the theorem follows.

It is well known that the maximum number $N(C)$ of non-overlapping translates of a given convex body $C \subseteq E^d$ so that they can touch C is not greater than $3^d - 1$ (see [25], [29]). On the other hand, it is clear that $Z(C) \leq N(C)$. Therefore $H(C) \leq 3^d - 1$ for all centrally symmetric convex bodies. If $d > 11$, Zong's bound is weaker than Rogers' bound mentioned above. However, $Z(C)$ seems to be much smaller than $N(C)$, perhaps $Z(C) \leq 2^d$.

The reader interested in a more complete account of the 'classical' part of Hadwiger's problem is referred to the books [8], [9] and the survey [4]. Here we mention only one recent result. Any convex body $C \subseteq E^3$ can be illuminated by at most two lines not intersecting C (see [5]).

Next we deal with a quantitative generalization of Hadwiger's problem due to Bezdek [3]. What is the least positive real number $f(d)$ such that for every o-symmetric convex body $C \subseteq E^d$ with distance function d_C and for every $l > f(d)$ there exist exterior points $p_1, \dots, p_n$ which illuminate C and $\sum_{i=1}^{n} d_C(o, p_i) = l$. It is shown in [3] that $f(2) = 6$. This problem looks very hard even for smooth o-symmetric convex bodies. In this case one can conjecture that for every $l > d(d+1)$ there exist exterior points $p_1, \dots, p_{2d}$ which illuminate C and $\sum_{i=1}^{2d} d_C(o, p_i) = l$. It is easy to see that the conjecture is true for $l > 2d^2$.

One can extend Hadwiger's problem for more than one convex body as well. Let $C_1, \dots, C_n \subseteq E^d$ be a family of n disjoint convex bodies. What is the minimum number of points in the exterior of $C_1 \cup \dots \cup C_n$ needed to illuminate (according to the definition of Soltan [46]) every boundary point of $C_1 \cup \dots \cup C_n$. This problem is open even for $d = 2$ and $n = 3$. The only known result in this direction is due to Soltan [49]. Any $n \geq 3$ convex bodies in the plane can be illuminated by at most $2n + 1$ directions (according to the definition of Boltyanski [7]).

The last problem in this survey was inspired by a result of Soltan [47]. Let $C \subseteq E^d$ be a convex body. A set $S \subseteq \mathrm{bd}C$ is called an inner illumination system for C if and only if for every $q \in \mathrm{bd}C$ there exists a point $p \in S$ different from q such that the open line segment $]p, q[$ is completely contained in the interior of C. Soltan proved in [47] that every convex body $C \subseteq E^d$ contains an inner illumination system consisting of at most $d+1$ points (for a different proof, see [27]). There is an interesting variant of this problem for convex polytopes. A convex polytope P is said to be illuminated by its vertices if and only if for every vertex u of P there is another vertex v of P such that the open line segment $]u, v[$ is contained in $\mathrm{int}P$. Let $k(d)$ be the smallest positive integer k such that there exists a d-dimensional convex polytope P of k vertices with the property that P is illuminated by its vertices. Hadwiger asked whether $k(d)$ equals $2d$, i.e. the number of vertices of the d-dimensional cross-polytope, for all d. Mani proved in [39] that $k(d) = 2d$, for $d \leq 7$, whereas, for large d, the approximate value of $k(d)$ is $d + 2\sqrt{d}$.

In connection with this problem Grünbaum posed in [27] the following conjecture. If no proper subset of an inner illumination system S of a convex body $C \subseteq E^d$ can illuminate $\mathrm{bd}C$ (in other words the illumination system is primitive), then $|S| \leq 2^d$. It is easy to see that the conjecture is true for $d = 2$. Recently Soltan [51] confirmed the conjecture for $d = 3$ and, even more recently, Boltyanski, Martini and Soltan [10] disproved the conjecture for $d \geq 4$ by constructing convex bodies containing arbitrary large primitive inner illumination systems.

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