Regular sphere packings

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Abstract

A collection of non-overlapping spheres in the space is called a packing. Two spheres are said to be neighbours if they have a boundary point in common. A packing is called k-regular if each sphere has exactly k neighbours. We are concerned with the following question. What is the minimum number of not necessarily congruent spheres

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which may form a k-regular packing? In general, for which natural numbers n and k does there exist a connected k-regular packing of exactly n spheres?

1. Introduction

A collection of non-overlapping spheres in the space is called a packing. Two spheres are said to be neighbours if they have a boundary point in common. A packing is called k-regular if each sphere has exactly k neighbours. Let $T(k)$ denote the minimum number of congruent spheres which may form a k-regular packing. Obviously, $T(1) = 2$, $T(2) = 3$, and $T(3) = 4$. Arrangements obtained by placing the centres of the spheres in the vertices of a regular octahedron and a regular icosahedron show that $T(4) \leq 6$ and $T(5) \leq 12$. It is not difficult to prove that $T(4) = 6$. The equality $T(5) = 12$ was established by Fejes Tóth and Harborth [2]. Wegner showed by an example that $T(6) \leq 240$ (see [2]). The problem of finiteness of $T(7)$ and $T(8)$ remains open. Kertész [3] proved that $T(k) = \infty$ for the remaining values of $k \leq 12$. We note that the maximum number of non-overlapping unit spheres which can touch one unit sphere is 12 (see [6]) which implies that k-regular unit sphere packings do not exist for $k \geq 13$.

Here we consider a similar question. What is the minimum number $B(k)$ of not necessarily congruent spheres which may form a k-regular packing? In general, for which natural numbers n and k does there exist a connected k-regular packing of exactly n spheres? It is easy to see that finite k-regular packings with not necessarily congruent spheres also do not exist for $k \geq 13$. However, a 12-regular packing with incongruent spheres can be obtained by a suitable chosen stereographic projection from the surface $\mathbb{S}^3$ of the unit sphere in $\mathbb{E}^4$ into $\mathbb{E}^3$ of the 120 spheres inscribed in the faces of a regular spherical 120-cell on $\mathbb{S}^3$ (cf. [1]) .

In this paper we determine all possible numbers of spheres which may form a connected k-regular packing for $k \leq 6$. We also give bounds on $B(k)$ for $7 \leq k \leq 10$. We note that the problem of finiteness of $B(11)$ remains open.

2. Preliminary lemmas

With a packing of spheres one can associate a graph, called the nerve of the packing. The vertices of this graph correspond to the spheres and two vertices are joined by an edge if and only if the corresponding spheres are neighbours. According to a celebrated theorem of Koebe [4], every planar graph is the nerve of some circle packing in the plane. Combining this result with a classical theorem of Kuratowski [5] we obtain the following characterization of nerves of circle packings. A graph is a nerve of some circle packing if and only if it contains no subdivision of the complete graph K_5 or the complete bipartite graph $K_{3,3}$. However, a similar characterization of nerves of sphere packings is not known. In this chapter we give some necessary conditions for nerves of sphere packings, in the sequel called space nerves, for short. We start with a well known lemma. For the sake of completeness we include its easy proof as well.

Lemma 1. *The complete graph K_6 is not a subgraph of a space nerve.*

Proof. Suppose, for contradiction, that there exist six mutually touching spheres. Consider the images of these spheres under an inversion with centre at the common point of two of these spheres. The images of the two spheres incident to the centre of inversion are two parallel planes and thus the images of the other spheres are four mutually touching congruent spheres with coplanar centres which is impossible. □

We prove two similar statements for complete bipartite graphs.

Lemma 2. *If the complete bipartite graph $K_{3,6}$ is a subgraph of a space nerve then the subgraphs induced by the two vertex classes of $K_{3,6}$ are cycles of lengths three and six.*

Proof. Suppose that $K_{3,6}$ is a subgraph of a space nerve. Let S_1, S_2, S_3 and $S_4, S_5, S_6, S_7, S_8, S_9$ denote the spheres corresponding to the vertices of the three-element and the six-element vertex classes of $K_{3,6}$, respectively. Consider the intersections C_1, C_2, C_3 of the spheres S_1, S_2, S_3 with the plane of their centres. Let C' and C'' be the two circles which simultaneously touch C_1, C_2, C_3 and do not separate them. Applying a suitable inversion, if necessary, we may assume that C' and C'' are indeed circles. Here one of C' and C'' may contain C_1, C_2, C_3. The circles C' and C'' are disjoint otherwise S_1, S_2, S_3 cannot have six common neighbours. Indeed, if $C' \cap C'' \neq \emptyset$ then an inversion whose centre is a common point of C' and C'' transforms S_1, S_2, S_3 to inspheres of a circular cone or cylinder which cannot have more than two common neighbours. It is known that for any two non intersecting circles there exists an inversion which carries them into concentric circles. Applying such an inversion to C' and C'' the images of S_1, S_2, S_3 under this inversion are congruent spheres whose centres are not collinear. Therefore we may assume that S_1, S_2, S_3 are congruent unit spheres whose centres are not collinear and thus the centres of $S_4, S_5, S_6, S_7, S_8, S_9$ lie on the line l going through the centre of the circle determined by the centres of S_1, S_2, S_3

and being perpendicular to the plane of this circle. We may also assume, applying an inversion whose centre is a common point of S_4 and l, if necessary, that S_4 is a plane. Without loss of generality we may also assume that S_4 is horizontal, $S_1, S_2, S_3, S_5, S_6, S_7, S_8, S_9$ are above S_4, and the distances of the centres of S_5, S_6, S_7, S_8, S_9 from S_4 in this order form an increasing sequence. Let S'_5, S'_6, S'_7, S'_8 denote the spheres above S_4 whose centres lie on l, the distances of their centres from S_4 are $\frac{1}{3}, \frac{5}{6}, \frac{7}{6}, \frac{5}{3}$, and their radii are $\frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{3}$, respectively. Note that S'_5, S'_6, S'_7, S'_8 touch S_1, S_2, S_3 if S_1, S_2, S_3 touch each other. Now, the highest point of S_5 is above S'_5 and thus S_6 is above the lowest point of S'_6. The highest point of S_6 is above S'_6 otherwise S_6 cannot touch S_1, S_2, S_3 since $S_6 \subset S'_6$. Repeating this argument for S_7 and S_8 we obtain that the distance of the highest point of S_8 from S_4 is at least 2 with equality if and only if S_1, S_2, S_3 mutually touch each other, $S_5 = S'_5$, $S_6 = S'_6$, $S_7 = S'_7$, $S_8 = S'_8$, and thus S_9 is the other common tangent plane of S_1, S_2, S_3. From this the lemma follows. $\square$

The proof of Lemma 2 implies the following.

Corollary 1. *The complete bipartite graph $K_{3,7}$ is not a subgraph of a space nerve.*

Lemma 3. *If the complete bipartite graph $K_{4,4}$ is a subgraph of a space nerve then the subgraphs induced by the two vertex classes of $K_{4,4}$ are cycles of length four.*

Proof. Suppose that $K_{4,4}$ is a subgraph of a space nerve. Let S_1, S_2, S_3, S_4 and S_5, S_6, S_7, S_8 denote the spheres corresponding to the vertices of the two vertex classes of $K_{4,4}$. As in the proof of Lemma 2 we may assume that S_1, S_2, S_3 are congruent unit spheres whose centres are not collinear and thus the centres of S_5, S_6, S_7, S_8 lie on the line l going through the centre of the circle of the centres of S_1, S_2, S_3 and being perpendicular to the plane of this circle. If S_4 is not a unit sphere whose centre lies on the above circle then at most two of S_5, S_6, S_7, S_8 can touch all S_1, S_2, S_3, S_4 since a line can intersect a two-sheeted hyperboloid or a plane not containing the line in at most two points. Thus S_1, S_2, S_3, S_4 are unit spheres whose centres are cocircular. We may assume, applying an inversion whose centre is a common point of S_5 and l, if necessary, that S_5 is a plane. Without loss of generality we may also assume that S_5 is horizontal, $S_1, S_2, S_3, S_4, S_6, S_7, S_8$ are above S_5, and the distances of the centres of S_6, S_7, S_8 from S_5 in this order form an increasing sequence. Let S'_6, S'_7 denote the spheres above S_5 whose centres lie on l, the distances of their centres from S_5 are $\frac{1}{2}, \frac{3}{2}$, respectively, and their radius is $\frac{1}{2}$. Note that S'_6, S'_7 touches S_1, S_2, S_3, S_4 if the centres of S_1, S_2, S_3, S_4 form a square of sidelength 2. Now, by a similar argument as in the proof of Lemma 2 we obtain that the distance of the highest point of S_7 from S_5 is at least 2 with equality if and only if the centres of S_1, S_2, S_3, S_4 form a square of sidelength 2, and thus $S_6 = S_6'$, $S_7 = S_7'$, and S_8 is the other common tangent plane of S_1, S_2, S_3, S_4. From this the lemma follows. □

The proof of Lemma 3 implies the following.

Corollary 2. *The complete bipartite graph $K_{4,5}$ is not a subgraph of a space nerve.*

3. Low-regular sphere packings

In this section we determine all possible numbers n and $k \leq 6$ for which there exist connected k-regular packings of n spheres. For $k \leq 5$ it is known (see e.g. [7]) that there exist connected k-regular packings of n circles for

(1) $k = 1$ and $n = 2$,
(2) $k = 2$ and $n \geq 3$,
(3) $k = 3$ and $n \geq 4$ where n is even,
(4) $k = 4$ and $n \geq 6$ where $n \neq 7$,
(5) $k = 5$ and $n \geq 12$ where $n \neq 14$ and n is even.

This immediately implies the existence of k-regular packings of n spheres in the cases listed above. Furthermore, it is easy to construct k-regular packings of n spheres for $(k, n) = (4, 5), (4, 7), (5, 8), (5, 10), (5, 14)$:

For $k = 4$ and $n = 5$ consider four unit spheres and a sphere of radius $\sqrt{3/2} - 1$ whose centres coincide with the vertices and the centre of a regular tetrahedron of sidelength 2, respectively.

For $k = 4$ and $n = 7$ consider four unit spheres S_1, S_2, S_3, S_4 and a sphere S_5 of radius $1/2$ whose centres coincide with the vertices and the centre of a rectangle of sidelengths 2 and $\sqrt{5}$, respectively. Furthermore, consider two distinct spheres S_6 and S_7 of radius 2 which touches all S_1, S_2, S_3, S_4. Note that S_5 is disjoint from S_6 and S_7.

For $k = 5$ and $n = 8$ consider five unit spheres S_1, S_2, S_3, S_4, S_5 whose centres form the vertices of a regular pentagon of sidelength 2. Let S_6 be the sphere whose centre coincide with the centre of this regular pentagon and which touches S_1, S_2, S_3, S_4, S_5. Furthermore, consider two distinct spheres S_7 and S_8 of radius 3 which touches all S_1, S_2, S_3, S_4, S_5. Note that S_6 is disjoint from S_7 and S_8.

For $k = 5$ and $n = 10$ consider eight unit spheres whose centres coincide with the vertices of a square based antiprism whose edgelengths are 2 (cf. [1]).

Let S' and S'' be the two parallel planes enclosing these eight spheres in such a way that each plane touches four of them, corresponding to the square faces of the antiprism. Finally, apply an inversion with centre in between S' and S'' and lying outside the eight spheres.

For $k = 5$ and $n = 14$ start with a 5-regular packing of 8 spheres (see above) and choose a sphere S of this packing. Now consider the union of this packing and its image under the inversion with respect to S deleting the sphere S.

Now, taking into account the facts that 6 spheres cannot form a 5-regular packing and that an odd number of spheres cannot form a k-regular packing for $k = 3, 5$ we obtain

Theorem 1. *For $k \leq 5$ there exist connected k-regular packings of n spheres if and only if*

(1) $k = 1$ *and* $n = 2$,
(2) $k = 2$ *and* $n \geq 3$,
(3) $k = 3$ *and* $n \geq 4$ *where n is even,*
(4) $k = 4$ *and* $n \geq 5$,
(5) $k = 5$ *and* $n \geq 8$ *where n is even.*

Next we determine all possible numbers of spheres which may form a connected 6-regular packing.

Theorem 2. *There exists a connected 6-regular packing of n spheres if and only if $n \geq 8$ and $n \neq 9$.*

Proof. Consider a 6-regular sphere packing and let G be its space nerve. Then G has at least 7 vertices. If G has 7 vertices then G is the complete graph K_7 which is impossible by Lemma 1. Thus G has at least 8 vertices.

First we show that the number of vertices of G cannot be 9. Suppose, for contradiction, that G has 9 vertices. Then the complement G' of G is a 2-regular graph, i.e., G' is the union of disjoint cycles. If G' contains a cycle of length 3 then G contains a $K_{3,6}$ and the vertices of the three-element vertex class of this $K_{3,6}$ are disconnected in G which is impossible by Lemma 2. If G' contains a cycle of length 4 then G contains a $K_{4,5}$ which is impossible by Corollary 2. Thus G' is a cycle of length 9. Let $\{S_1, S_2, \ldots, S_9\}$ be a sphere packing whose space nerve is G. By a suitable choice of the indices, we may assume that $S_i \cap S_{i+1} = \emptyset$ for $1 \leq i \leq 8$, and $S_1 \cap S_9 = \emptyset$ as well. Consider the images $S'_1, S'_2, \ldots, S'_9$ of the spheres $S_1, S_2, \ldots, S_9$, respectively, under an inversion with centre at $S_1 \cap S_3$. Now S'_1 and S'_3 are parallel planes, and S'_5, S'_6, S'_7, S'_8 are congruent spheres touching both S'_1 and S'_3. Since S'_2 touches all S'_5, S'_6, S'_7, S'_8 the centres of S'_5, S'_6, S'_7, S'_8 are cocircular. This implies that

S_4' touches S_5' since S_4' touches S_6', S_7', S_8'. This contradiction shows that G cannot have 9 vertices.

Now, we turn to the constructions. First we show that there exists a 6-regular packing of $4m$ spheres for all $m \geq 2$.

Consider four mutually tangent unit spheres S_1, S_2, S_3, S_4. Let O denote the centre of the tetrahedron whose vertices are the centres of S_1, S_2, S_3, S_4. Let S_5, S_6, S_7, S_8 be the homothetic images of S_1, S_2, S_3, S_4, respectively, with centre O and ratio $-(3+2\sqrt{2})$. Then the sphere S_5 touches S_2, S_3, S_4, the sphere S_6 touches S_1, S_3, S_4, the sphere S_7 touches S_1, S_2, S_4, and the sphere S_8 touches S_1, S_2, S_3. It is clear that the spheres $S_1, S_2, \ldots, S_8$ form a 6-regular packing.

To construct a 6-regular packing of 12 spheres first replace the spheres S_5, S_6, S_7, S_8 with the spheres S_5', S_6', S_7', S_8' concentric with S_5, S_6, S_7, S_8, respectively, whose radius is less than the radius of S_5, S_6, S_7, S_8 by a suitable small positive ε and translate S_5', S_6', S_7', S_8' so that the sphere S_5' touches S_2, S_3, S_4 and is disjoint from S_1, the sphere S_6' touches S_1, S_3, S_4 and is disjoint from S_2, the sphere S_7' touches S_1, S_2, S_4 and is disjoint from S_3, and the sphere S_8' touches S_1, S_2, S_3 and is disjoint from S_4. Let $S_9, S_{10}, S_{11}, S_{12}$ be four mutually touching spheres disjoint from S_1, S_2, S_3, S_4 such that the sphere S_9 touches S_6', S_7', S_8', the sphere S_{10} touches S_5', S_7', S_8', the sphere S_{11} touches S_5', S_6', S_8', and the sphere S_{12} touches S_5', S_6', S_7'. It is easy to check that if ε is small enough then $S_1, S_2, S_3, S_4, S_5', S_6', S_7', S_8', S_9, S_{10}, S_{11}, S_{12}$ form a 6-regular packing.

Repeating the above argument one can easily construct a 6-regular packing of $4m$ spheres for all $m \geq 3$.

Next we show that there exist a 6-regular packing of $4m+2$ spheres for all $m \geq 2$.

Consider six unit spheres $S_1, S_2, \ldots, S_6$ whose centres form a regular octahedron of side length two where the centres of S_i and S_{i+3} are opposite vertices of the octahedron, $i = 1, 2, 3$. Let S_7, S_8, S_9, S_{10} be four congruent spheres of radius $4+\sqrt{14}$ disjoint from the octahedron such that the sphere S_7 touches S_1, S_2, S_3, the sphere S_8 touches S_1, S_5, S_6, the sphere S_9 touches S_3, S_4, S_5, and the sphere S_{10} touches S_2, S_4, S_6. Note that S_7, S_8, S_9, S_{10} are mutually touching spheres. It is easy to check that these 10 spheres form a 6-regular packing.

To construct a 6-regular packing of 14 spheres first replace the spheres S_7, S_8, S_9, S_{10} with the spheres $S_7', S_8', S_9', S_{10}'$ concentric with S_7, S_8, S_9, S_{10}, respectively, whose radius is less than the radius of S_7, S_8, S_9, S_{10} by a suitable small positive ε and translate $S_7', S_8', S_9', S_{10}'$ so that the sphere S_7' touches S_1, S_2, S_3 and is disjoint from S_4, S_5, S_6, the sphere S_8' touches S_1, S_5, S_6 and is disjoint from S_2, S_3, S_4, the sphere S_9' touches S_3, S_4, S_5 and is disjoint from S_1, S_2, S_6, and the sphere S'_{10} touches S_2, S_4, S_6 and is disjoint from S_1, S_3, S_5. Let $S_{11}, S_{12}, S_{13}, S_{14}$ be four mutually touching spheres disjoint from $S_1, S_2, \ldots, S_6$ such that the sphere S_{11} touches S'_7, S'_8, S'_9, the sphere S_{12} touches S'_7, S'_8, S'_{10}, the sphere S_{13} touches S'_7, S'_9, S'_{10}, and the sphere S_{14} touches S'_8, S'_9, S'_{10}. It is easy to check that if ε is small enough then $S_1, S_2, \ldots, S_6, S'_7, S'_8, S'_9, S'_{10}, S_{11}, S_{12}, S_{13}, S_{14}$ form a 6-regular packing.

Repeating the above argument one can easily construct a 6-regular packing of $4m+2$ spheres for all $m \geq 3$.

Now, we turn to the case when the number of spheres is odd. First we construct a 6-regular packing of 11 spheres. Let S_1, S_2, S_3, S_4 four mutually tangent unit spheres. Let O denote the the centre of the tetrahedron whose vertices are the centres of S_1, S_2, S_3, S_4. Furthermore, let P denote the point of the line segment joining O with the centre of S_4 whose distance from O is a suitable small positive ε', say $\varepsilon' = 0.001$. Let S_5, S_6, S_7 be the homothetic images of S_1, S_2, S_3, respectively, with centre P and with ratio greater than 1 such that S_i touches S_{i+4} for $i = 1, 2, 3$. Let S_8 be the sphere which touches S_2, S_3, S_4, S_6, S_7 and is disjoint from S_1. Let S_9 be the sphere which touches S_1, S_3, S_4, S_5, S_7 and is disjoint from S_2. Let S_{10} be the sphere which touches S_1, S_2, S_4, S_5, S_6 and is disjoint from S_3. Finally, let S_{11} be the sphere which touches $S_5, S_6, \ldots, S_{10}$ and which is disjoint from S_1, S_2, S_3. We note that the spheres S_4 and S_{11} are disjoint. It is easy to check that these 11 spheres form a 6-regular packing.

Observe that if we start with four congruent spheres S_1, S_2, S_3, S_4 whose centres form the vertices of a regular tetrahedron of edge length 2 and whose radius is $1 - \varepsilon''$ where ε'' is a suitable small positive number then repeating the above procedure we obtain a sphere packing $\mathcal{S}$ in which S_1, S_2, S_3, S_4 have three neighbours and $S_5, S_6, \ldots, S_{11}$ have six neighbours.

Now take a 6-regular packing constructed above which consists of $2m$ spheres, $m \geq 6$. Remove the outermost four spheres of this packing. Let S'_1, S'_2, S'_3, S'_4 denote the outermost four spheres of the packing of the remaining $2m-4$ spheres. Recall that S'_1, S'_2, S'_3, S'_4 are mutually disjoint congruent spheres whose centres form the vertices of a regular tetrahedron. If we choose ε and ε'' appropriately we can apply a suitable similarity transformation for this packing which transforms the spheres S'_1, S'_2, S'_3, S'_4 to the spheres S_1, S_2, S_3, S_4 of $\mathcal{S}$. It is easy to check that in this way together with the spheres $S_5, S_6, \ldots, S_{11}$ we obtain a 6-regular packing of $2m+3$ spheres.

To complete the proof it remains to construct a 6-regular packing of 13 spheres.

Let S_1, S_2, S_3 be three mutually tangent unit spheres. Let S_4, S_5, S_6 be the images of S_1, S_2, S_3, respectively, under the translation orthogonal to the plane of the centres of S_1, S_2, S_3 and of length $2+\varepsilon$ where ε is a suitable small positive number, say $\varepsilon = 0.001$. Let S_7, S_8, S_9 be three mutually tangent congruent spheres such that the triple S_7, S_8, S_9 is homothetic with the triple S_1, S_2, S_3 with ratio greater than 1, the sphere S_7 touches S_1, S_4, the sphere S_8 touches S_2, S_5, and the sphere S_9 touches S_3, S_6. Let S_{10} be the sphere which touches $S_1, S_2, S_4, S_5, S_7, S_8$ and whose radius is smaller than 4. Let S_{11} be the sphere which touches $S_1, S_3, S_4, S_6, S_7, S_9$ and whose radius is smaller than 4. Let S_{12} be the sphere which touches $S_2, S_3, S_5, S_6, S_8, S_9$ and whose radius is smaller than 4. Finally, let S_{13} be the sphere which touches $S_1, S_2, \ldots, S_6$. It is easy to check that if ε is small enough then these 13 spheres form a 6-regular packing.

This completes the proof of Theorem 2. $\square$

4. High-regular sphere packings

In this section we give bounds for the minimum number of spheres which can form a k-regular packing for $7 \le k \le 10$.

Theorem 3. *$B(7) = 12$ or 14.*

Proof. Consider a 7-regular sphere packing and let G be its space nerve. Then G has at least 8 vertices. If G has 8 vertices then G is the complete graph K_8 which is impossible by Lemma 1. Thus G has at least 9 vertices. Note that the number of vertices of G is even since G is 7-regular and the number of vertices with odd degree is even in any graph.

First we show that the number of vertices of G cannot be 10. If G has 10 vertices then the complement G' of G is a 2-regular graph, i.e., G' is the union of disjoint cycles. If G' contains a cycle of length 3 then G contains a $K_{3,7}$ which is impossible by Corollary 1. If G' contains a cycle of length 4 or 5 then G contains a $K_{4,5}$ which is impossible by Corollary 2. Thus G' is a cycle $v_1 v_2 \cdots v_{10}$. Then G contains a $K_{4,4}$ with vertex classes $\{v_1, v_2, v_3, v_4\}$ and $\{v_6, v_7, v_8, v_9\}$. By Lemma 3, the vertices v_1, v_2, v_3, v_4 lie on a cycle in G, a contradiction. This implies that $B(7) \ge 12$.

To complete the proof it remains to construct a 7-regular packing of 14 spheres. Let S_1, S_2, S_3 be three mutually touching congruent spheres of radius $\lambda = 1 + \varepsilon$ where ε is a suitable small positive number, say $\varepsilon = 0.001$. Let O_1, O_2, O_3 denote the centres of S_1, S_2, S_3, respectively, and let O denote the centre of the triangle $O_1 O_2 O_3$. Let S_4, S_5, S_6 be three congruent spheres of radius $7 + 4\sqrt{3}$ such that the sphere S_{i+3} touches S_i and its centre lies on the half line emanating at O and going through O_i for $i = 1, 2, 3$. Then S_4, S_5, S_6 are mutually disjoint. Let S_7, S_8, S_9 be three mutually touching congruent spheres such that the sphere S_7 touches S_2, S_3, S_5, S_6, the sphere S_8 touches

S_1, S_3, S_4, S_6, and the sphere S_9 touches S_1, S_2, S_4, S_5. Let S_{10} be the sphere disjoint from S_1, S_2, S_3 which touches $S_4, S_5, S_6, S_7, S_8, S_9$. Furthermore, let $S_{11}, S_{12}, S_{13}, S_{14}$ be the images of S_7, S_8, S_9, S_{10} under the reflection with respect to the plane spanned by O_1, O_2, O_3. Now, if we increase continuously the radius λ of S_1, S_2, S_3 then we will find a value λ_0 at which S_{10} and S_{14} become parallel planes. Applying an inversion whose centre is between S_{10} and S_{14} and lies outside the spheres $S_1, S_2 \dots , S_{12}$ to $S_1, S_2, \dots , S_{14}$ it is easy to check that we obtain a 7-regular packing of 14 spheres. □

Theorem 4. *$B(8) \leq 20$.*

Proof. Let S_1, S_2, S_3 be three mutually touching spheres with centres at $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1 - \varepsilon)$, respectively, where the radius of S_1 and S_2 is $1/\sqrt{2}$ and ε is a suitable small positive number, say $\varepsilon = 0.001$. Let S_4 be the sphere which touches S_1, S_2, S_3, and the two coordinate planes containing the third coordinate axis, and whose centre is separated from the origin by the plane spanned by the centres of S_1, S_2, S_3. Let S_5 and S_6 be two touching congruent spheres whose centres lie on the first and the second coordinate axis, respectively, which touch S_4, and which are different from S_1 and S_2, respectively. Let S_7 be the sphere whose centre lies on the third axis, which touches S_4, S_5, S_6, and whose centre is separated from the origin by the centre of S_3. Finally, let $\mathcal{S}$ be the family of the images of $S_1, S_2, \dots , S_7$ under the symmetry group generated by the reflection with respect to the three coordinate planes. It is easy to check that $\mathcal{S}$ is an 8-regular packing of 20 spheres. □

Theorem 5. *$B(9) \leq 64$.*

Proof. Let S_1, S_2, S_3 be three mutually touching congruent spheres with centres at $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$, respectively. Let r denote the radius of the sphere which touches S_1, S_2, S_3, and all three coordinate planes and whose centre is separated from the origin by the plane spanned by the centres of S_1, S_2, S_3. Let S_4 be the sphere which touches S_1, S_2, S_3, whose centre is separated from the origin by the plane spanned by the centres of S_1, S_2, S_3, and whose radius $\lambda = r - \varepsilon$ where ε is a suitable small positive number, say $\varepsilon = 0.001$. Let S_5 be the sphere whose centre lies on the first axis and which touches S_2, S_3, S_4. Let S_6 be the sphere whose centre lies on the second axis and which touches S_1, S_3, S_4. Let S_7 be the sphere whose centre lies on the third axis and which touches S_1, S_2, S_4. Let S_8 be the sphere whose centre lies on the first axis, is separated from the origin by the centre of S_5, and which touches S_5, S_4. Let S_9 be the sphere whose centre lies on the second axis, is separated from the origin by the centre of S_6, and which touches S_6, S_4. Let

S_{10} be the sphere whose centre lies on the third axis, is separated from the origin by the centre of S_7, and which touches S_7, S_4. It is easy to see that the spheres S_8, S_9, S_{10} are mutually disjoint. Now, if we decrease continuously the radius λ of S_4 then we will find a value λ_0 at which S_8, S_9, S_{10} mutually touch each other. Let $S_{11}, S_{12}, \ldots, S_{20}$ be the images of $S_1, S_2, \ldots, S_{10}$ under the inversion with respect to the sphere of radius $1/\sqrt{2}$ around the origin. Finally, let $\mathcal{S}$ be the family of the images of $S_1, S_2, \ldots, S_{20}$ under the symmetry group generated by the reflection with respect to the three coordinate planes. It is easy to check that $\mathcal{S}$ is a 9-regular packing of 64 spheres. □

Theorem 6. $B(10) \leq 76$.

Proof. Let D be a regular dodecahedron of side length 2. Let O be the centre of D and let $P_1, P_2, \ldots, P_5$ be the vertices of a face of D. Let $S_1, S_2, \ldots, S_5$ be the unit sphere with centre $P_1, P_2, \ldots, P_5$, respectively. Let S_6 be the sphere which touches $S_1, S_2, \ldots, S_5$ and the triangular faces of the pyramid with base $P_1P_2 \ldots P_5$ and apex O. Let $S_7, S_8, \ldots, S_{12}$ be the images of $S_1, S_2, \ldots, S_6$ under the inversion with respect to the sphere with centre O and radius $\overline{OP_1} + 1$. Let S_{13} be the sphere which touch all $S_1, \ldots, S_5, S_7, \ldots, S_{11}$. It is easy to see that S_{13} is disjoint from S_6, S_{12} and do not intersect the boundary of the convex cone generated by the half lines starting at O and going through $P_1, P_2, \ldots, P_5$, respectively. Finally, let $\mathcal{S}$ be the family of the images of $S_1, S_2, \ldots, S_{13}$ under the symmetry group of the dodecahedron D. It is easy to check that $\mathcal{S}$ is a 10-regular packing of 76 spheres. □

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