Relative Newton numbers of regular polygons with equal side lengths

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Abstract

Let K_n denote a regular n-gon with unit side length. For two regular polygons K_n and K_m let the relative Newton number $N(K_n, K_m)$ of K_n and K_m be the maximum number of congruent copies of K_m that can touch K_n without having interior points in common. In this paper we determine the exact value of $N(K_n, K_m)$ for various pairs (n, m).

1. Introduction

The Newton number $N(K)$ of a convex body K is the maximum number of congruent copies of K that can touch K without having interior points in

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common. Even in the plane, the question for $N(K)$ is far from being trivial for most convex discs K. The exact value of $N(K)$ is known only for some special convex discs K, e.g. for regular n–gons [1, 5, 7, 8, 10, 11, 12], for Reuleaux triangles [2], for certain isosceles triangles [2, 6, 10], and for certain rectangles [5]. Also, some estimates are given for more general classes of convex discs involving different parameters of the discs, e.g. diameter, width, etc. [2, 4, 9, 10].

In this paper we consider the following generalization of the Newton number problem due to Wegner [10] (see also [3]). For two convex discs K and L, let the relative or parasitic Newton number $N(K, L)$ of K and L be the maximum number of congruent copies of L that can touch K without having interior points in common. Here we concentrate on the cases when K and L are (different) regular polygons with equal side lengths. Let K_n denote a regular n-gon with unit side length. We will determine the exact value of $N(K_n, K_m)$ for various pairs (n, m).

2. A general upper bound

In this section we give a simple general upper bounds for $N(K_n, K_m)$. For $n \geq 3$, let r_n and R_n denote the radius of the incircle and the circumcircle of K_n, respectively. Let K be a congruent copy of K_n and let L_1 and L_2 be two nonoverlapping congruent copies of K_m which touch K. Let O, O_1, O_2 be the centre of K, L_1, L_2, respectively. Then $\overline{O_1O_2} \geq 2r_m$ since L_1 and L_2 do not overlap. Furthermore, $r_n + r_m \leq \overline{OO_i} \leq R_n + R_m$ since L_i touches K, $i = 1, 2$. To estimate $N(K_n, K_m)$ from above it is enough to give a lower bound φ for the angle $\measuredangle O_1OO_2$. Without loss of generality we may assume that $\overline{OO_1} \leq \overline{OO_2}$. Now, an elementary argument shows that $\measuredangle O_1OO_2$ is not smaller than the smaller one of φ_1 and φ_2 where φ_1 corresponds to the case where $\overline{O_1O_2} = r_n + r_m$ and $\overline{OO_1} = \overline{OO_2} = R_n + R_m$, while φ_2 corresponds to the case where $\overline{OO_2} = R_n + R_m$ and $\overline{OO_1} = \overline{O_1O_2} = r_n + r_m$. Moreover, equality may occur only in these two cases.

Setting $\varphi = \min\{\varphi_1, \varphi_2\}$ we obtain the upper bound $N(K_n, K_m) \leq \lfloor 2\pi/\varphi \rfloor$. The next tables show the numerical data for $3 \leq n \leq 6$.

m	3	4	5	6	$7, 8, 9, 10$	$11, \ldots, 24$	$25, \ldots$
$N(K_3, K_m) \leq$	∞	7	6	5	4	3	2

m	3	4	5	$6, 7$	$8, \ldots, 12$	$13, \ldots, 29$	$30, \ldots$
$N(K_4, K_m) \leq$	21	8	6	5	4	3	2

m	3	4	5	6	7, 8	9, ..., 13	14, ..., 35	36, ...
$N(K_5, K_m) \leq$	20	9	7	6	5	4	3	2

m	3	4	5	6	7	8, 9, 10	11, ..., 16	17, ..., 41	42, ...
$N(K_6, K_m) \leq$	21	10	8	7	6	5	4	3	2

For $(n, m) = (6, 10)$ the previous argument gives the upper bound $N(K_6, K_{10}) \leq 5$. More precisely, we obtain $\measuredangle O_1 O O_2 \geq \frac{2\pi}{5}$ with equality if and only if $\overline{O_1 O_2} = 2r_{10}$ and $\overline{OO_1} = \overline{OO_2} = R_6 + R_{10}$. However, it is impossible to arrange one hexagon and two decagons in such a way that the two decagons touch the hexagon without overlapping each other and for the centers O, O_1, O_2 of the hexagon and the two decagons, respectively, the above conditions hold. Thus $\measuredangle O_1 O O_2 > \frac{2\pi}{5}$ from which $N(K_6, K_{10}) \leq 4$.

Combining the above upper bounds with the constructions given in Figures 1-4 we obtain the exact value of $N(K_n, K_m)$ in various cases (the easy calculations showing the correctness of the constructions are left to the reader).

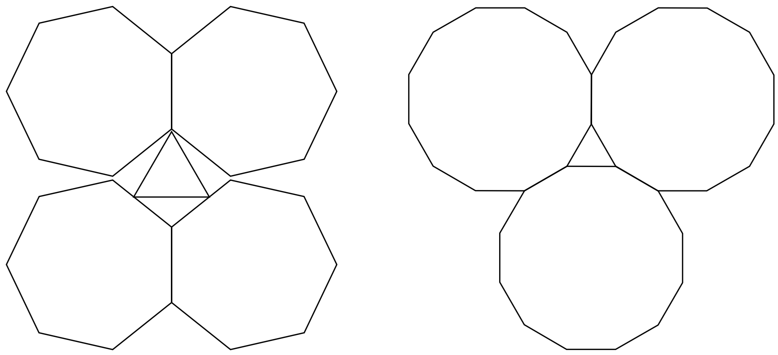

Figure 1: $N(K_3, K_7) \geq 4$ and $N(K_3, K_m) \geq 3$ for $m = 11, 12$.

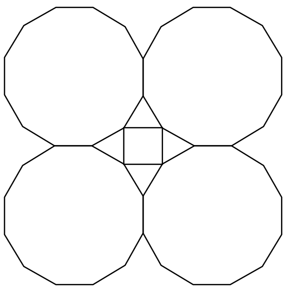

Figure 2: $N(K_4, K_m) \geq 4$ for $m = 8, 9, 10, 11, 12$.

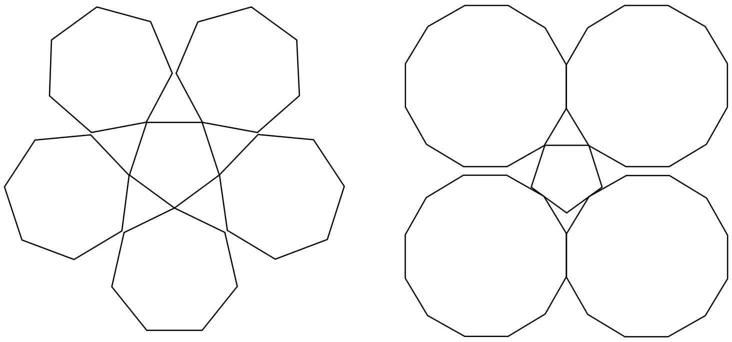

Figure 3: $N(K_5, K_7) \geq 5$ and $N(K_5, K_m) \geq 4$ for $m = 9, 10, 11, 12$.

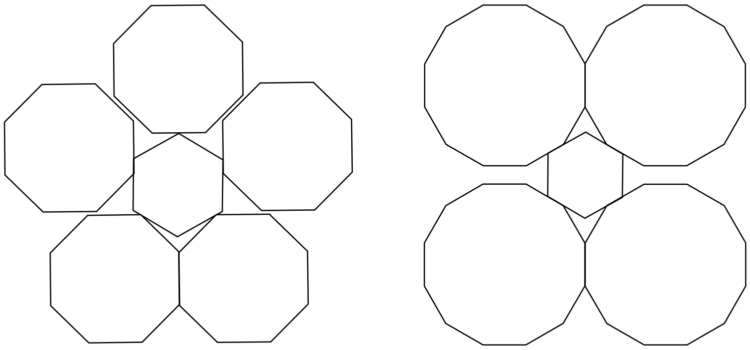

Figure 4: $N(K_6, K_8) \geq 5$ and $N(K_6, K_m) \geq 4$ for $m = 10, 11, 12$.

Theorem 1.

$$\begin{aligned}
N(K_3, K_7) &= 4, \\
N(K_3, K_m) &= 3, \quad m = 11, 12, \\
N(K_4, K_m) &= 4, \quad m = 8, 9, 10, 11, 12, \\
N(K_5, K_7) &= 5, \\
N(K_5, K_m) &= 4, \quad m = 9, 10, 11, 12, \\
N(K_6, K_8) &= 5, \\
N(K_6, K_m) &= 4, \quad m = 10, 11, 12.
\end{aligned}$$

3. $N(K_6, K_3)$

In this section we consider the case $(n, m) = (6, 3)$ and we prove

Theorem 2. $N(K_6, K_3) = 18$.

Proof. Figure 5 shows that $N(K_6, K_3) \geq 18$.

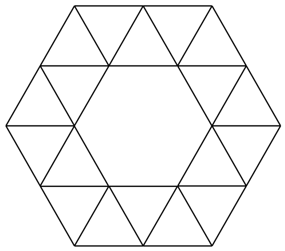

Figure 5: $N(K_6, K_3) \geq 18$.

To prove that $N(K_6, K_3) \leq 18$ consider a regular hexagon H with unit side length and its homothetic image H' where the centre of homothety coincides with the centre of H and the ratio of the homothety is $\frac{3}{2}$. The distance between the corresponding parallel edges of H and H' is $\frac{\sqrt{3}}{4}$. Divide the boundary F of H' into 18 segments of equal length by 18 posts so that at each vertex of F there is a post. The distance between any two consecutive posts is $\frac{1}{2}$.

Now, we introduce the following scoring system. A regular triangle T with unit side length touching H scores 1 point for each post in its interior, and $\frac{1}{2}$ point for each post on its boundary.

Consider a collection $\mathcal{C}$ of nonoverlapping regular triangles with unit side length touching H. If a post is in the interior of one of the triangles, then this post cannot be in the interior or on the boundary of any other triangle of $\mathcal{C}$. Such a post contributes exactly 1 point to the total score. If a post is on the boundary of a triangle, then it can be on the boundary of at most one other triangle and it cannot be in the interior of any triangle of $\mathcal{C}$. Therefore, such a post can contribute at most 1 point to the total score. Since the number of posts is 18, the total score is at most 18.

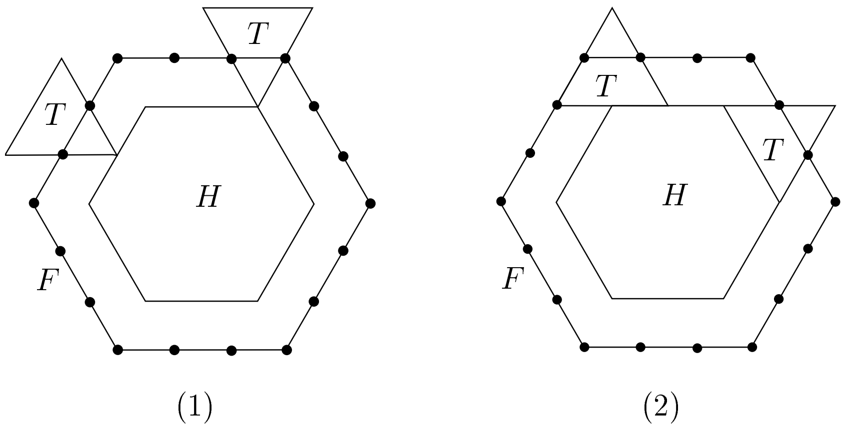

Figure 6: Two cases in the proof of Theorem 2.

To complete the proof it remains to show that any regular triangle T of $\mathcal{C}$ touching H scores at least 1 point. If a triangle T of $\mathcal{C}$ contains a post in its interior it scores at least 1 point. On the other hand, if T does not contain a post in its interior, then we distinguish two cases. If $T \cap H$ is a point, then the boundary of T must intersect F at two posts (see Figure 6(1)), i.e. T scores $2 \times \frac{1}{2} = 1$ point. If $T \cap H$ is a line segment, then the boundary of T contains at least two posts (see Figure 6(2)), and thus T scores at least 1 point again. $\square$

4. $N(K_3, K_6)$

In this section we consider the case $(n, m) = (3, 6)$ and we prove

Theorem 3. $N(K_3, K_6) = 4$.

Proof. Figure 7 shows that $N(K_3, K_6) \geq 4$.

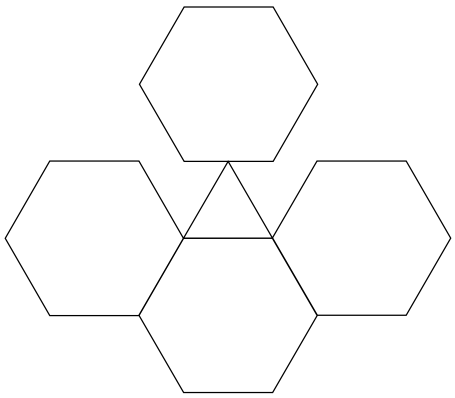

Figure 7: $N(K_3, K_6) \geq 4$.

To prove that $N(K_3, K_6) \leq 4$ consider the unit triangle T with vertices $T_1(-\frac{1}{2}, 0)$, $T_2(\frac{1}{2}, 0)$, $T_3(0, \frac{\sqrt{3}}{2})$ in the coordinate system and the boundary F of the convex hexagon with vertices $F_1(-\frac{7}{8}, -\frac{3\sqrt{3}}{8})$, $F_2(\frac{7}{8}, -\frac{3\sqrt{3}}{8})$, $F_3(\frac{5}{4}, 0)$, $F_4(\frac{3}{8}, \frac{7\sqrt{3}}{8})$, $F_2(-\frac{3}{8}, \frac{7\sqrt{3}}{8})$, $F_6(-\frac{5}{4}, 0)$. The angles of F are $\frac{2\pi}{3}$ and the lengths of the edges of F are $\frac{7}{4}$ and $\frac{3}{4}$ alternately, $\frac{15}{2}$ in total. Let H be a regular hexagon with unit side length which touches T. Let $H_1, H_2, \ldots, H_6$ denote the vertices of H in the clockwise order. First we show that the length $|F \cap H|$ of the part of F covered by H is never smaller than $\frac{3}{2}$ and we also determine the cases when equality occurs. This will be done by a case-by-case analysis based on the position of H with respect to the position of T.

Case 1. $T \cap H$ is a line segment. Without loss of generality we may assume that $T \cap H$ lies on $\overline{H_1H_2}$ and $\overline{T_1T_2}$. By symmetry, we may also assume that H_1 is on the right of T_1. For $0 \le t \le 1$ let $H(t)$ denote the position of H where the distance between H_1 and T_1 is t. Then $|F \cap H(t)| = \frac{7}{4}$ for $0 \le t < \frac{3}{4}$ and $|F \cap H(t)| > |F \cap H(1)| = \frac{3}{2}$ for $\frac{3}{4} \le t < 1$.

Case 2. $T \cap H$ is a common vertex of T and H. Without loss of generality we may assume that T_2 and H_2 coincide. By symmetry, we may also assume that $\measuredangle T_1T_2H_1 \le \frac{\pi}{2}$. For $0 \le \varphi \le \frac{\pi}{2}$ let $H(\varphi)$ denote the position of H where $\measuredangle T_1T_2H_1 = \varphi$. It is clear that $|F \cap H(\varphi)| = |F \cap H(\frac{\pi}{2})| = \frac{3}{2}$ for $\frac{\pi}{3} \le \varphi < \frac{\pi}{2}$. The next range we deal with is $\arcsin \frac{3\sqrt{3}}{8} \le \varphi < \frac{\pi}{3}$. The angle $\varphi_0 = \arcsin \frac{3\sqrt{3}}{8}$ here corresponds to the position of H where H_1 is incident to $\overline{F_1F_2}$. Using the notations of Figure 8 the triangles $T_2F_3F^I$ and $T_2F^{III}F^{IV}$ are congruent thus applying the angle bisector theorem for the triangle $F^{II}T_2F^{IV}$ with the angle bisector T_2F^{III} we obtain $|\overline{F^{II}F^{III}}| > |\overline{F^{III}F^{IV}}| = |\overline{F^IF_3}|$. This implies that $|F \cap H(\varphi)| > |F \cap H(\frac{\pi}{3})| = \frac{3}{2}$ for $\varphi_0 \le \varphi < \frac{\pi}{3}$.

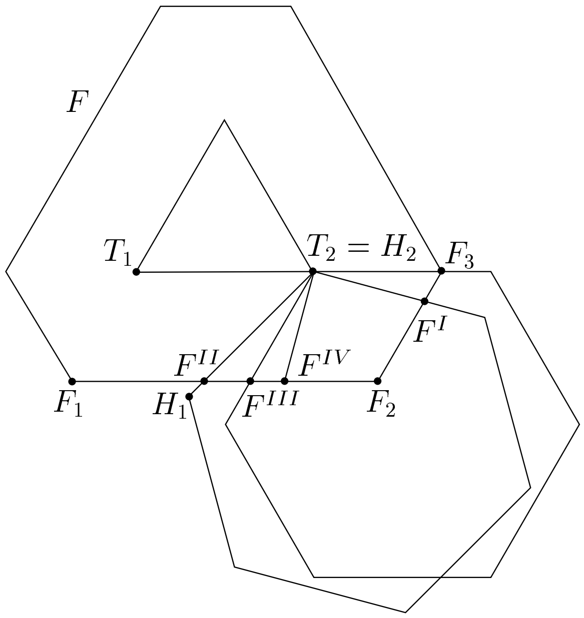

Figure 8: Case 2 in the proof of Theorem 3 with $\arcsin \frac{3\sqrt{3}}{8} \le \varphi < \frac{\pi}{3}$.

To prove that $|F \cap H(\varphi)| > \frac{3}{2}$ holds for the remaining range $0 < \varphi < \varphi_0$ as well consider the particular angles $\varphi_1 = \frac{\pi}{6}$, $\varphi_2 = 0.346866$, $\varphi_3 = 0.168302$. Easy calculation shows that $|F \cap H(\varphi_{i-1}) \cap H(\varphi_i)| > \frac{3}{2}$, $i = 1, 2, 3$, and $|F \cap H(\varphi)| > |\overline{F_1F_2} \cap H(\varphi_3)| > \frac{3}{2}$ for $0 < \varphi \le \varphi_3$. Thus $|F \cap H(\varphi)| > \frac{3}{2}$ for $0 < \varphi < \varphi_0$.

Case 3. $T \cap H$ is a vertex of T. Without loss of generality we may assume that T_2 lies on $\overline{H_1H_2}$. By symmetry, we may also assume that $\measuredangle T_1T_2H_1 \leq \frac{\pi}{3}$. For $0 \leq \varphi \leq \frac{\pi}{3}$ and $0 \leq t \leq 1$ let $H(\varphi, t)$ denote the position of H where $\measuredangle T_1T_2H_1 = \varphi$ and the distance between H_2 and T_2 is t. Fix φ and consider the particular values t_1, t_2, t_3 of t where t_1 corresponds to the position of H where H_1 lies on $\overline{F_1F_2}$, t_2 corresponds to the position of H where $\overline{H_2H_3}$ goes through F_3, and t_3 corresponds to the position of H where H_2 lies on $\overline{F_3F_4}$. If t_1 is not defined set $t_1 = 0$. This situation occurs when $\varphi < \arcsin \frac{3\sqrt{3}}{8}$.

We know from Case 2 that $|F \cap H(\varphi, 0)| \geq \frac{3}{2}$ and $|F \cap H(\varphi, 1)| \geq \frac{3}{2}$. Thus $|F \cap H(\varphi, t)| > \frac{3}{2}$ for $0 < t \leq t_1$ and $t_3 \leq t < 1$ since $F \cap H(\varphi, 0) \subset F \cap H(\varphi, t)$ for $0 < t \leq t_1$ and $F \cap H(\varphi, 1) \subset F \cap H(\varphi, t)$ for $t_3 \leq t < 1$.

Next we deal with the range $t_1 \leq t \leq t_2$. Using the notations of Figure 9 consider the triangles H_1GI and $H_2'H_2J$ where H_1G and JH_2' are parallel to F_1F_2 and F_2F_3, respectively, and $\overline{H_1I} = \overline{H_1H_1'} = \overline{H_2H_2'}$. Since $\measuredangle H_1IG = \measuredangle H_2'H_2J = \frac{\pi}{3}$ and $\measuredangle H_1GI = \measuredangle H_2'JH_2$ therefore these triangles are congruent and thus $\overline{H_1G} = \overline{H_2'J}$. This implies that $|F \cap H(\varphi, t)| = |F \cap H(\varphi, t_1)| > \frac{3}{2}$ for $t_1 \leq t \leq t_2$.

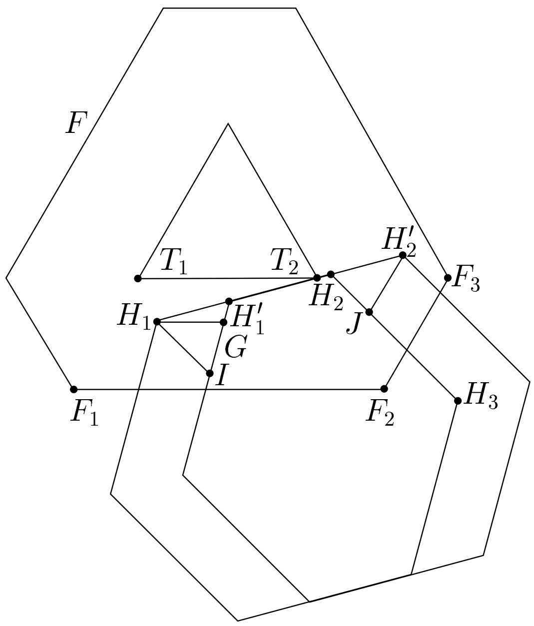

Figure 9: Case 3 in the proof of Theorem 3 with $t_1 \leq t \leq t_2$.

Finally, let $t_2 \leq t \leq t_3$. Using the notations of Figure 10 consider the triangles F^IIF^{II} and F_3JF^{III}. In these triangles $\overline{F^IF^{II}} \leq \overline{F^II} = \overline{JF^{III}} \leq \overline{F^{III}F_3}$ since $\measuredangle F^IIF^{II} = \measuredangle F_3JF^{III} = \frac{\pi}{3}$ and $\measuredangle F^IF^{II}I \geq \frac{\pi}{3} \geq \measuredangle JF_3F^{III}$. Hence $|F \cap H(\varphi, t)| \geq |F \cap H(\varphi, t_2)| > \frac{3}{2}$ for $t_2 < t \leq t_3$ as well.

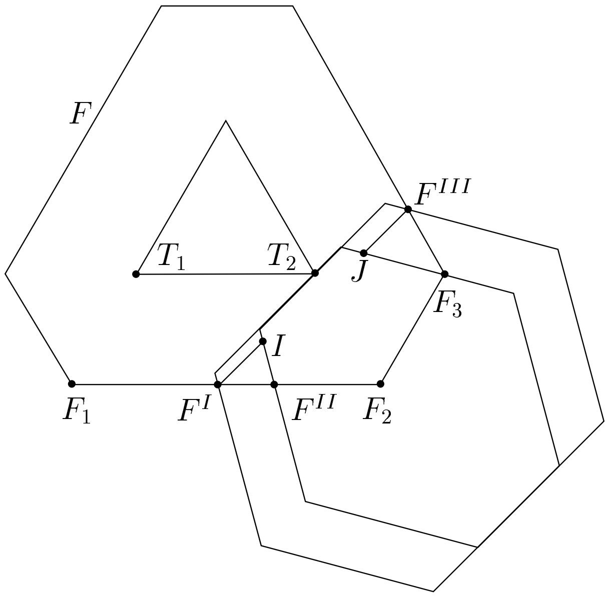

Figure 10: Case 3 in the proof of Theorem 3 with $t_2 \le t \le t_3$.

Case 4. $T \cap H$ is a vertex of H. Without loss of generality we may assume that H_1 lies on $\overline{T_1T_2}$. By symmetry, we may also assume that the distance between T_1 and H_1 is at least $\frac{1}{2}$. For $0 \le \varphi \le \frac{\pi}{3}$ and $\frac{1}{2} \le t \le 1$ let $H(\varphi, t)$ denote the position of H where $\measuredangle T_1H_1H_6 = \varphi$ and the distance between T_1 and H_1 is t.

First we show that $|F \cap H(\varphi, \frac{1}{2})| > \frac{3}{2}$ for $0 \le \varphi \le \frac{\pi}{3}$. By symmetry, it is enough to prove this for $\frac{\pi}{6} \le \varphi \le \frac{\pi}{3}$. Consider the particular angles $\varphi_1 = \frac{\pi}{6}$, $\varphi_2 = \arcsin \frac{3\sqrt{3}}{8}$, $\varphi_3 = 0.980525$, and $\varphi_4 = \frac{\pi}{3}$. Easy calculation shows that $|F \cap H(\varphi_i, \frac{1}{2}) \cap H(\varphi_{i+1}, \frac{1}{2})| > \frac{3}{2}$, $i = 1, 2, 3$. Thus $|F \cap H(\varphi, \frac{1}{2})| > \frac{3}{2}$ for $\frac{\pi}{6} \le \varphi \le \frac{\pi}{3}$.

Now fix φ and consider the particular values t_1, t_2, t_3, t_4 of t where the value t_1 corresponds to the position of H where $\overline{H_1H_2}$ goes through the point F^I of $\overline{F_1F_2}$ whose distance from F_2 is $\frac{1}{4}$, the value t_2 corresponds to the position of H where $\overline{H_5H_6}$ goes through F_1, the value t_3 corresponds to the position of H where F_2 lies on the boundary of H, and the value t_4 corresponds to the position of H where H_2 lies on $\overline{F_2F_3}$. If t_1 is not defined set $t_1 = \frac{1}{2}$. This situation occurs when $\varphi > \frac{\pi}{3} - \arctan \frac{3\sqrt{3}}{5}$. If t_2 is not defined set $t_2 = t_1$. If t_3 is not defined or $t_2 > t_3$ set $t_3 = t_2$. If t_4 is not defined set $t_4 = t_3$.

First let $\frac{1}{2} \le t \le t_1$ and suppose that $\frac{1}{2} < t_1$. Then F_1 lies inside $H(\varphi, t_1)$. Indeed, this is clear when $H(\varphi, \frac{1}{2})$ contains the point F^{II} of F_1F_2 on the left of F_1 whose distance from F_1 is $\frac{1}{4}$ since $t_1 \le \frac{3}{4}$. On the other hand, if $H(\varphi, \frac{1}{2})$ does not contain F^{II} then easy calculation shows that $\varphi > \arcsin\frac{2}{3} - \frac{\pi}{6}$ from which $t_1 < 0.54406$. Combining this fact with the fact that $H(\varphi, \frac{1}{2})$ contains the point of F_1F_2 on the left of F_1 whose distance from F_1 is 0.2 the assertion follows. Thus $|F \cap H(\varphi, t_1)| > \overline{F_1F^I} = \frac{3}{2}$. Now using the notations of Figure 11 the inequality $\overline{H^IF^{III}} < \overline{F^{III}F^{IV}}$ implies that $|F \cap H(\varphi, t)| \geq |F \cap H(\varphi, t_1)| > \frac{3}{2}$.

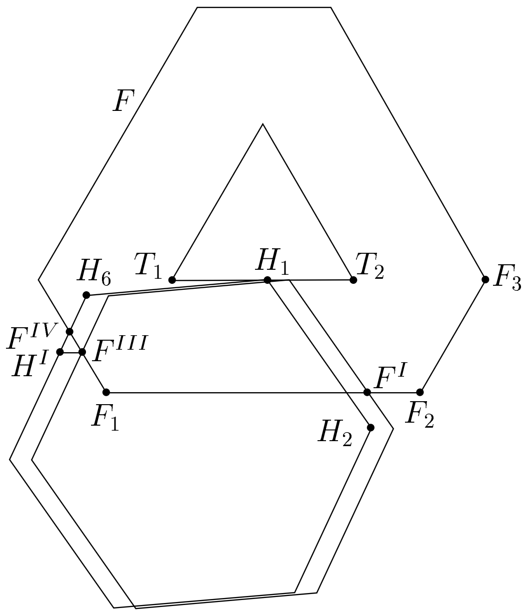

Figure 11: Case 4 in the proof of Theorem 3 with $\frac{1}{2} \leq t \leq t_1$.

Secondly, let $t_1 \leq t \leq t_2$ and suppose that $\frac{1}{2} < t_2$. Then $\overline{F_1F^I} \subset H(\varphi, t_1)$ and thus $|F \cap H(\varphi, t)| > \frac{3}{2}$.

Thirdly, let $t_2 \leq t \leq t_3$ and suppose that $t_2 < t_3$. Then $|F \cap H(\varphi, t)| = |F \cap H(\varphi, t_2)| > \frac{3}{2}$.

Finally, let $t_3 \leq t < 1$. We know from Case 2 that $|F \cap H(\varphi, 1)| \geq \frac{3}{2}$ thus using similar argument as for the range $\frac{1}{2} < t < t_1$ above we obtain $|F \cap H(\varphi, t)| > |F \cap H(\varphi, t_3)| > \frac{3}{2}$ for $t \leq t_4$ and $|F \cap H(\varphi, t)| > |F \cap H(\varphi, 1)| \geq \frac{3}{2}$ for $t > t_4$.

This finishes the proof of the inequality $|F \cap H| \geq \frac{3}{2}$. In addition, we obtained that $|F \cap H| = \frac{3}{2}$ if and only if $T \cap H$ is a common vertex of T and H, and the angle of the half lines emanating from $T \cap H$ and going through the centres of T and H, respectively, is not smaller than $\frac{5\pi}{6}$. This implies that at most four copies of H can touch T without overlapping each other. $\square$

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