Regular circle packings

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1. Introduction

A collection of nonoverlapping circles in the plane is called a packing. Two circles are said to be neighbours if they possess a boundary point in common. A packing is called a k-neighbour packing if each circle has exactly k neighbours and it is called connected if the union of the circles is connected. In this paper we consider the following problem. For which natural numbers n and k can we construct a connected k-neighbour packing consisting of exactly n circles. The following theorem answers our question.

Theorem 1. *There exists a k-neighbour packing of n circles if and only if one of the following holds:*

(1) $k = 1$ *and* $n = 2$,

(2) $k = 2$ *and* $n \geqslant 3$,

(3) $k = 3$ *and* $n \geqslant 4$, *where* n *is even,*

(4) $k = 4$ *and* $n \geqslant 6$, *where* $n \neq 7$,

(5) $k = 5$ *and* $n \geqslant 12$, *where* $n \neq 14$ *and* n *is even.*

For some additional results concerning packings with given number of neighbours we refer the reader to the survey paper [3] by G. Fejes Tóth and W. Kuperberg.

With a packing of circles one can associate a planar graph, called the nerve of the packing, whose vertices are the centres of the circles and two vertices are joined by an edge if the corresponding circles are neighbours. According to a celebrated theorem of Koebe [6], rediscovered by Andreev [1, 2] and Thurston [8], every planar graph is the nerve graph of some circle packing. Therefore, since the nerve graph of a k-neighbour packing is k-regular, our theorem is equivalent with the proposition that a connected k-regular planar graph with n vertices exists for and only for pairs of k and n satisfying one of the conditions (1)-(5) in Theorem 1. To my knowledge, this result is not explicitly stated in the literature. On the other hand, as Stanislav Jendrol kindly pointed out to me, it follows from different known results. It is easy to see that n has to be even if k is odd since the number of vertices with odd degree is even in any graph. Furthermore, as a trivial consequence of Euler's theorem, a simple planar graph with n vertices has at most $3n - 6$ edges. Since in a k-regular planar graph with n vertices the number of edges is $\frac{kn}{2}$ we obtain $n(6-k) \geqslant 12$, which shows that $k \leqslant 5$ and $n \geqslant 4$, if $k = 3$, $n \geqslant 6$, if $k = 4$, $n \geqslant 12$, if $k = 5$. A theorem of Malkevitch [7] implies that there are no 4-regular and 5-regular planar graphs having only triangles and exactly one quadrangle, i.e. there are no 4-regular and 5-regular planar graphs with $n = 7$ and $n = 14$ vertices, respectively. Thus relations (1)-(5) form necessary conditions for the existence of a connected k-regular planar graph with n vertices. The fact that these conditions are also sufficient follows from different constructions given by Grünbaum [5] p.282 and Fisher [4].

In this paper we give a simple alternative proof to the special cases $k = 4$, $n = 7$ and $k = 5$, $n = 14$ of Malkevitch's theorem. We also prove the existence part of our theorem by giving direct constructions of circle packings for all pairs k and n described by relations (1)-(5).

2. Exclusion of the cases $k = 4$, $n = 7$ and $k = 5$, $n = 14$

As we have already pointed out, first we give a simple proof for two special cases of Malkevitch's theorem.

Lemma 1. *There is no 4-regular planar graph with 7 vertices.*

Proof. For a contradiction assume that there exists a 4-regular planar graph G with 7 vertices. From Euler's formula it can be shown that G divides the plane into nine regions, eight triangles and one quadrangle. Without loss of generality we may assume that the quadrangle (a, b, c, d) is a bounded region.

Then neither (a, c) nor (b, d) can be an edge of G. In fact, if, say, (a, c) is an edge of G, the remaining three vertices e, f and g must be interior to the regions (a, c, b) and (a, c, d). If all three vertices are interior to one region, the other region is left with a vertex of degree two. On the other hand, if two vertices are interior to one region, the vertex in the other region can be joined to at most three boundary vertices of its region.

Now, we show that the only 2-paths from a to c pass through either b or d. For a contradiction suppose that $((a, e), (e, c))$ is a 2-path from a to c and e is not equal to b or d. If the vertices f and g are in one of the regions (a, e, c, b) and (a, e, c, d), then the other region is left with a vertex of degree two. On the other hand, if the vertex f is in (a, e, c, b) and g is in (a, e, c, d), then f must be joined to all of the vertices a, e, c and b, and g must be joined to all of the vertices a, e, c and d. This leaves a and c with incidences of five. Similarly, we can show that the only 2-paths from b to d pass through either a or c.

Finally, consider the four "empty" triangles bounding the edges of the quadrangle (a, b, c, d) with respective third vertices e, f, g, h. The previous observations show that e, f, g, h must all be distinct, a contradiction. $\square$

Lemma 2. *There is no 5-regular planar graph with 14 vertices.*

Proof. For a contradiction assume that there exists a 5-regular planar graph G with 14 vertices. From Euler's formula it can be shown that G divides the plane into twenty-three regions, twenty-two triangles and one quadrangle. Without loss of generality we may assume that the quadrangle (a, b, c, d) is a bounded region.

The following elementary observation will be essential in our proof. Suppose that the region $(u, v, \ldots, z)$ contains s vertices. Then the total number of free degrees of u, v, $\ldots$, z must be at least 5 if $s = 1$, at least 8 if $s = 2$, at least 9 if $s = 3$, at least 8 if $s = 4$, and at least 7 if $s = 5$.

First, we prove that neither (a, c) nor (b, d) can be an edge of G. In fact, if (a, c) is an edge of G, the remaining ten vertices must be interior to the regions (a, c, b) and (a, c, d). Since the free degree of b is three and the total number of free degrees of the vertices a, b, c is seven, the region (a, c, b) contains at least five vertices. Similarly, the region (a, c, d) also contains at least five vertices. But the total number of free degrees of the vertices a, b, c, d is only ten, thus it is impossible to arrange five-five vertices in the regions (a, c, b) and (a, c, d).

Now, we show that the only 2-paths from a to c pass through either b or d. For a contradiction suppose that $((a, e), (e, c))$ is a 2-path from a to c and e is not equal to b or d. Then the remaining nine vertices must be interior to the regions (a, e, c, b) and (a, e, c, d). Since the free degrees of b and d are three, the regions (a, e, c, b) and (a, e, c, d) contain at least three-three vertices. We have two different cases.

Case 1. Exactly three vertices are interior to one region, say to (a, e, c, b). Let f, g, h be these three vertices. Then the vertices b, f, g, h must form a complete 4-graph. It is easy to see that one of the vertices f, g, h, say f, must be interior to the region determined by the other three vertices b, g, h. This leaves f with an incidence of three.

Case 2. Four vertices are interior to one region and five to the other. But the total number of free degrees of the vertices a, b, c, d, e is only thirteen.

Similarly, we can show that the only 2-paths from b to d pass through either a or c.

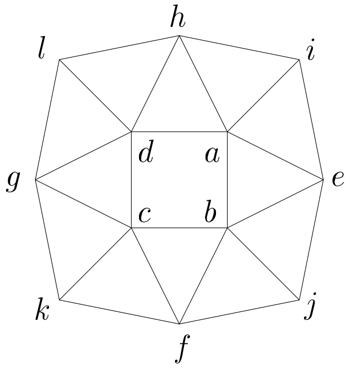

Figure 1.

Next, consider the four "empty" triangles bounding the edges of the quadrangle (a, b, c, d) with respective third vertices e, f, g, h. Obviously, the vertices e, f, g, h must all be distinct. The free degree of the vertices a, b, c, d is one, so let i, j, k, l be the other endpoints of the fifth edges incident to a, b, c and d, respectively. The above observations show that the vertices a, b, c, d, e, f, g, h and the vertices i, j, k, l are pairwise distinct, moreover $i \neq k$ and $j \neq l$. We prove that the vertices i, j, k, l are pairwise distinct, too. For a contradiction assume that two vertices, say i and j coincide. Since the free degrees of a and b are zero, the vertex i must be adjacent too. This leaves e with an incidence of three, a contradiction. Now, the free degrees of the vertices a, b, c, d are zero, thus the edge-pairs $((i, a), (a, e))$, $((e, b), (b, j))$, $\ldots$, $((h, a), (a, i))$ determine eight "empty" triangles (see Figure 1).

Finally, the unbounded region (i, e, j, f, k, g, l, h) must contain the remaining two vertices in its interior. Since the total number of free degrees of the vertices i, e, j, f, k, g, l, h is twelve, two of these vertices must be adjacent. Then we have five combinatorically different cases:

(1) i and j are adjacent,

(2) i and f are adjacent,

(3) i and k are adjacent,

(4) e and f are adjacent,

(5) e and g are adjacent.

The verification of the impossibility of these cases is simple and is left to the reader. □

3. Regular circle packing constructions

$k = 2$

This case is settled by a "ring" of n successively touching congruent circles whose centres form the vertices of a regular n-gon.

$k = 3$

It is easy to see that four circles can be arranged in the plane such that any two of them touch each other. On the other hand, for $n \geqslant 6$ and n is even, we consider two rings of circles in centrally similar position each containing $\frac{n}{2}$ circles such that the corresponding circles are neighbours.

$k = 4$

First, we deal with the case when n has the form $2m$, where $m \geqslant 3$. Obviously, a slight modification of the preceding construction results in a 4-neighbour packing consisting of n circles.

Next, we consider the case when $n = 4m - 1$, $m \geqslant 3$. For $n = 11$ and $n = 15$ see Figure 2 and Figure 3, respectively.

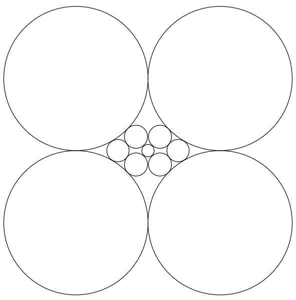

Figure 2.

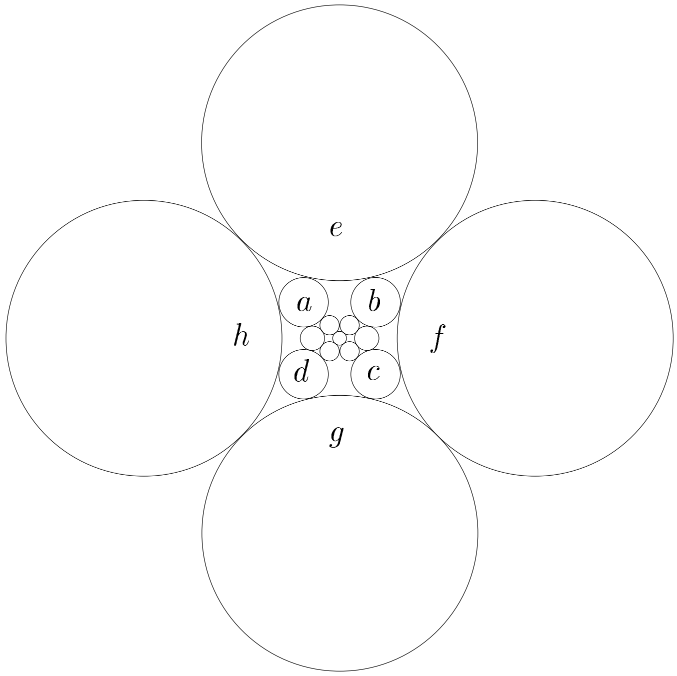

Figure 3.

In Figure 3 the circles a, b, c, d and the circles e, f, g, h are congruent and their centres form the vertices of two squares. For $n = 19$ consider the following construction. Decrease slightly the radii of the circles e, f, g, h and translate these circles so that the new circles e', f', g', h' touch a and b, b and c, c and d, d and a, respectively. Note that the centres of e', f', g', h' also form the vertices of a square. Finally, add four new congruent circles i, j, k, l so that

(1) i touches e', f', l, j,

(2) j touches f', g', i, k,

(3) k touches g', h', j, l,

(4) l touches h', e', k, i.

It is easy to see that all remaining numbers can be reached repeating the above procedure.

The case $n = 4m - 3$, where $m \geqslant 3$, can be settled similarly using the initial configurations $n = 9$, $n = 13$, and $n = 17$ (see Figures 4-6).

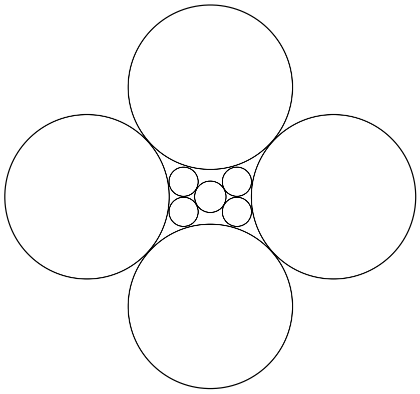

Figure 4.

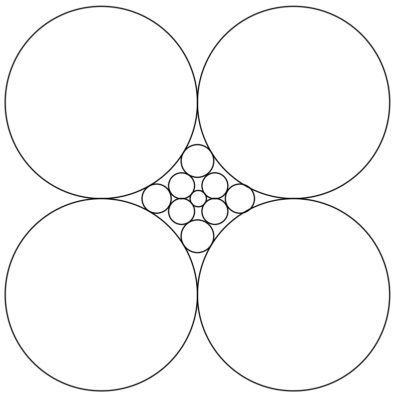

Figure 5.

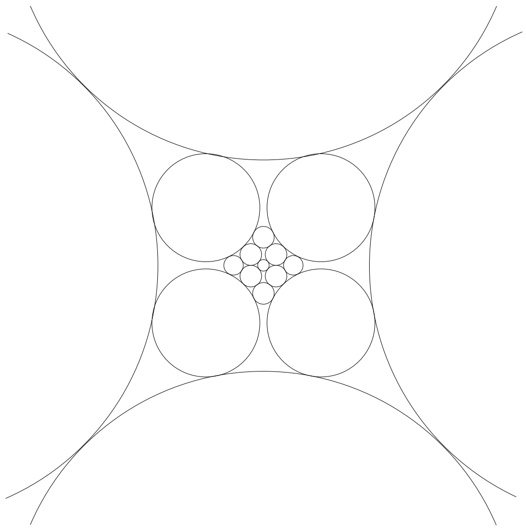

Figure 6.

$k = 5$

Let us consider, as a first step, the case $n = 4m$, where $m \geqslant 3$. In Figure 7 choose the angle at O to be equal to $\frac{8\pi}{n}$. We note that the existence of this arrangement follows by a simple continuity argument. Furthermore, if we rotate this figure around the point O by the multiples of $\frac{8\pi}{n}$, we obtain a 5-neighbour packing consisting of n circles.

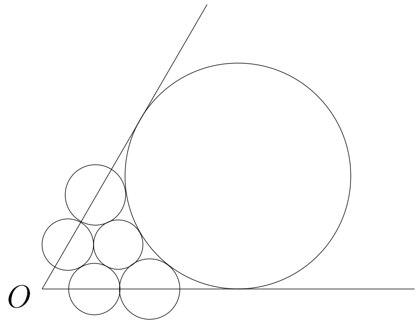

Figure 7.

To finish the proof we introduce the following pasting construction on 5-neighbour circle packings. Take a 5-neighbour packing consisting of n circles and choose a circle c of this collection. Let the new circle system be the union of the original circle packing and its image under the inversion corresponding to c deleting the circle c. This construction results in a 5-neighbour packing consisting of $2n - 2$ circles.

Now take a look at the case $n = 8m - 2$, where $m \geqslant 3$. Clearly, we can immediately reach these circle packings from the packings consisting of $n = 4m$ circles by the pasting construction.

Next, consider the case $n = 8m + 2$, where $m \geqslant 2$. In this case we need the initial configurations $n = 18$ and $n = 26$ (see Figure 8 and Figure 9). Then, one can show by induction that all these circle packings can be reached using the pasting construction, too.

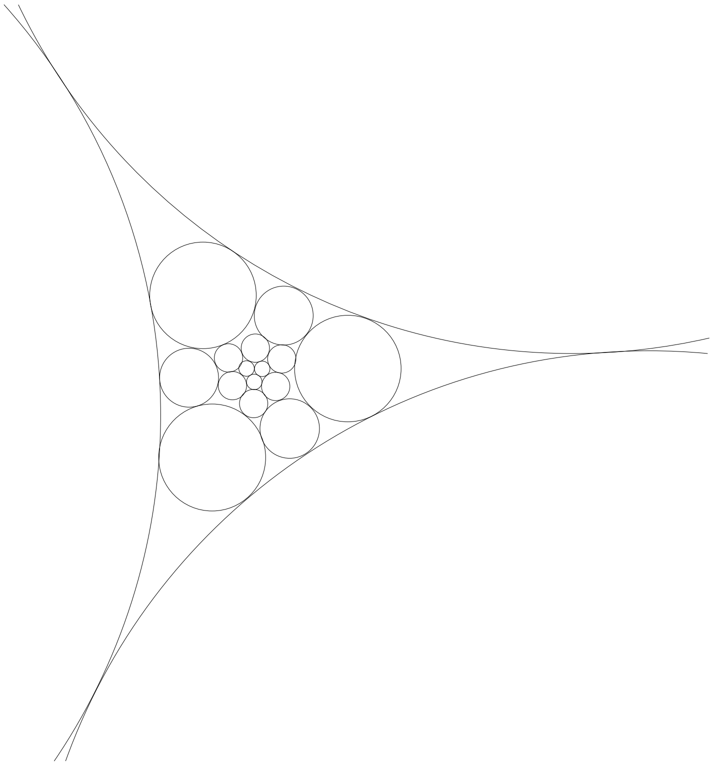

Figure 8.

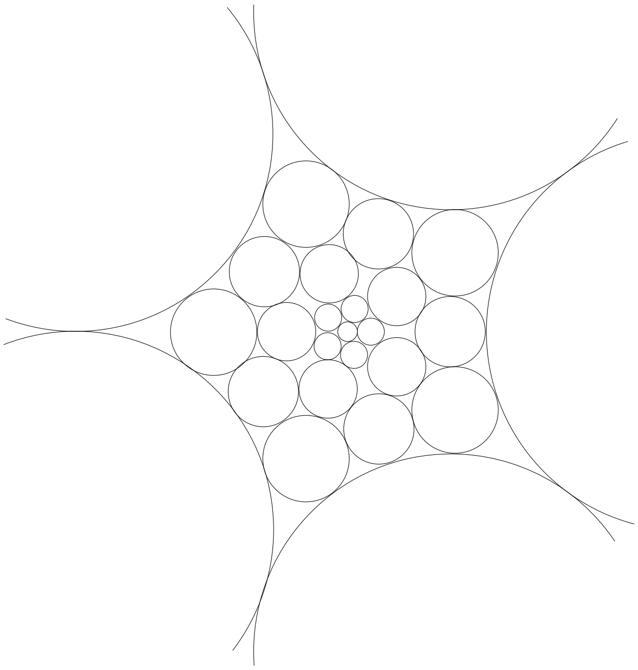

Figure 9.

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