Primitive illumination systems for families of convex bodies in the plane

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Abstract

Let $\mathcal{F} = \{C_1, C_2, \dots, C_n\}$ be a family of n disjoint convex bodies in the plane. We say that a set V of exterior light sources illuminates $\mathcal{F}$, if for every boundary point u of any member of $\mathcal{F}$ there is a point v in V such that u is visible from v, i.e. the open line segment joining u and v is disjoint from $\cup\mathcal{F}$. An illumination system V is called primitive if no proper subset of V illuminates $\mathcal{F}$. Let $p_{\max}(\mathcal{F})$ denote the maximum number of points forming a primitive illumination system for $\mathcal{F}$, and let $p_{\max}(n)$ denote the minimum of $p_{\max}(\mathcal{F})$ taken over all families $\mathcal{F}$ consisting of n disjoint convex bodies in the plane. The aim of this paper is to investigate the quantities $p_{\max}(\mathcal{F})$ and $p_{\max}(n)$.

1. Introduction

Let $\mathcal{F} = \{C_1, C_2, \ldots, C_n\}$ be a family of n disjoint convex bodies (i.e. compact convex sets with non-empty interior) in the plane. Following the terminology of László Fejes Tóth [4] (see also an earlier paper [6]), we say that a set V of exterior light sources illuminates $\mathcal{F}$, if for every boundary point u of any member of $\mathcal{F}$ there is a point v in V such that u is visible from v, i.e. the open line segment joining u and v is disjoint from $\cup\,\mathcal{F}$. We note the existence of a more restrictive definition of the illumination, see [1, 2].

An illumination system V for a family $\mathcal{F}$ of disjoint convex bodies is called *primitive* if no proper subset of V illuminates $\mathcal{F}$. Let $p_{\min}(\mathcal{F})$ and $p_{\max}(\mathcal{F})$ denote the minimum and the maximum number of points forming a primitive illumination system for $\mathcal{F}$ (set $p_{\max}(\mathcal{F}) = \infty$ if the maximum does not exist). Furthermore, let $p_{\min}(n)$ and $p_{\max}(n)$ denote the maximum and the minimum of $p_{\min}(\mathcal{F})$ and $p_{\max}(\mathcal{F})$, respectively, taken over all families $\mathcal{F}$ consisting of n disjoint convex bodies.

Fejes Tóth proved in [4] that $p_{\min}(n) = \max\{2n, 4n-7\}$. Recently Czyzowicz, Rivera-Campo and Urrutia [3] proved that $\lfloor \frac{4n+4}{3} \rfloor$ points are sufficient to illuminate n disjoint isothetic rectangles as well as n disjoint triangles.

The aim of this paper is to investigate the quantities $p_{\max}(\mathcal{F})$ and $p_{\max}(n)$.

2. Main Results

The first theorem below is a necessary condition for the finiteness of $p_{\max}(\mathcal{F})$.

Theorem 1. *Let $\mathcal{F} = \{C_1, C_2, \ldots, C_n\}$ be a family of n disjoint convex bodies in the plane. If $p_{\max}(\mathcal{F})$ is finite, then all the bodies $C_1, C_2, \ldots, C_n$ are convex polygons.*

The next theorem shows that without additional assumptions the converse of Theorem 1 is not true.

Theorem 2. *One can find a family $\{T_1, T_2, T_3\}$ of three disjoint triangles such that $p_{\max}(\{T_1, T_2, T_3\}) = \infty$.*

To give a sufficient condition for the finiteness of $p_{\max}(\mathcal{F})$ we need a definition. Let (C_1, C_2, C_3) be an ordered triple of elements of $\mathcal{F}$. We say that (C_1, C_2, C_3) is critical if there exists a line ℓ with the following properties:

(1) ℓ meets the bodies C_1, C_2, C_3 in this order in the line segments or points s_1, s_2, s_3,

(2) the open line segments on ℓ between s_1 and s_2 and between s_2 and s_3, respectively, are disjoint from $\cup\mathcal{F}$,

(3) the segments or points s_1 and s_2 are on the boundary of C_1 and C_2, respectively,

(4) ℓ separates C_1 and C_2,

(5) ℓ does not separate C_1 and C_3,

as shown in Figure 1.

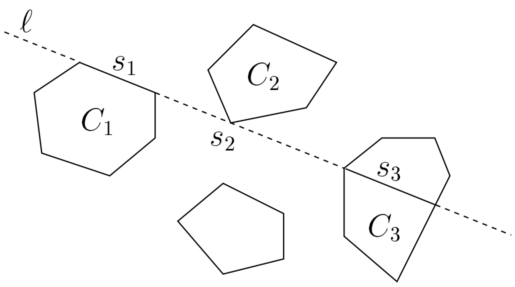


Figure 1

We call a line ℓ critical for (C_1, C_2, C_3) if it satisfies the properties (1)-(5) above.

Theorem 3. *Let $\mathcal{F} = \{C_1, C_2, \dots, C_n\}$ be a family of n disjoint convex polygons in the plane. If $\mathcal{F}$ contains no critical triple, then $p_{\max}(\mathcal{F})$ is finite.*

Surprisingly, the sufficient condition in Theorem 3 is not necessary.

Theorem 4. *One can find a family $\mathcal{F}$ of four disjoint convex polygons containing a critical triple, for which $p_{\max}(\mathcal{F})$ is finite.*

However, if $\mathcal{F}$ consists of only three polygons, the sufficient condition in Theorem 3 is almost necessary.

Theorem 5. *Let P_1, P_2, P_3 be three disjoint convex polygons which form a critical triple in this order. If there is a critical line ℓ for (P_1, P_2, P_3) separating P_2 and P_3, then $p_{\max}(\{P_1, P_2, P_3\}) = \infty$.*

After the proof of Theorem 5 we give a very complicated criterion, consisting of the list of 18 independent cases, for the finiteness of $p_{\max}(\{P_1, P_2, P_3\})$ for three convex polygons. Unfortunately, we are not able to give a compact criterion or to extend our results for the case of more than three polygons.

Next we will discuss the case of two convex polygons. According to Theorem 3, the maximum number of points forming a primitive illumination system for two convex polygons is always finite. In fact, the following stronger inequality holds.

Theorem 6. *If P_1 and P_2 are two disjoint convex polygons with k_1 and k_2 vertices, respectively, then $p_{\max}(\{P_1, P_2\}) \leqslant k_1 + k_2 - 1$.*

In the following theorem we determine the exact value of $p_{\max}(2)$.

Theorem 7. *If P_1 and P_2 are two disjoint convex polygons, then $p_{\max}(\{P_1, P_2\}) \geqslant 4$. On the other hand, one can find two disjoint triangles T_1, T_2 such that $p_{\max}(\{T_1, T_2\}) = 4$.*

Theorem 7 immediately implies

Corollary 1. $p_{\max}(2) = 4$.

Turning to the case $n = 3$ we prove

Theorem 8. *One can find a family $\{T_1, T_2, T_3\}$ of three disjoint triangles such that $p_{\max}(\{T_1, T_2, T_3\}) = 6$.*

Theorem 8 stimulates the following

Conjecture 1. $p_{\max}(3) = 6$.

Finally, we give an upper bound for $p_{\max}(n)$ in the case $n \geqslant 4$.

Theorem 9. *For any positive integer $n \geqslant 4$ one can find n disjoint triangles $T_1, T_2, \ldots, T_n$ such that $p_{\max}(\{T_1, T_2, \ldots, T_n\}) \leqslant 3n - 2$. This implies $p_{\max}(n) \leqslant 3n - 2$ for all $n \geqslant 4$.*

3. Proof of Theorem 1 and Theorem 2

To prove Theorem 1 we need some propositions.

Proposition 1. *Let C be a convex body in the plane and let P be a convex polygon circumscribed about C such that the vertices of P are exterior to C. If each side of P intersects the boundary of C in at most one point, then the vertices of P illuminate C.*

Proof. Let u be a boundary point of C. If u lies on a side of P, then the endpoints of this side illuminate u. On the other hand, if u is not on a side of P, then any supporting line of C at u cuts off at least one vertex of P which, clearly, illuminates u. □

Proposition 2. *For a convex body C other than a polygon there exist arbitrary large primitive illumination systems.*

Proof. Since C is not a polygon, it has infinitely many exposed points (recall that a point x on the boundary of C is called exposed if there exists a line l such that $C \cap l = \{x\}$). Choose k exposed points on the boundary of C, where k is a positive integer, and consider k supporting lines at these points. These lines determine a convex polygonal region circumscribed about C and having at most k sides. It is easy to see that one can choose the k points and the respective k supporting lines such that this polygon is bounded. By Proposition 1, the vertices of this polygon form an illumination system for C which is, obviously, primitive. □

Proposition 3. *If P is a convex polygon with k vertices, then $p_{\max}(\{P\}) = k$.*

Proof. Let $a_1, a_2, \ldots, a_k$ denote the sides of P in the clockwise order and consider a (not necessarily primitive) illumination system V for P. For $1 \leqslant i \leqslant k$, the open half plane bounded by the line containing the side a_i and disjoint from the polygon P must intersect V. By selecting only one illuminating point in each of these open half planes, we obtain an illumination system $V' \subseteq V$ for P consisting of at most k points. Therefore, the cardinality of any primitive illumination system of P is also at most k. On the other hand, one can easily construct a primitive illumination system for P consisting of exactly k points. □

Propositions 2 and 3 immediately imply Theorem 1 for the case of one convex body, see Proposition 4. We note that the higher dimensional analogue of this statement is also true, see [5].

Proposition 4. *Let C be a convex body in the plane. Then $p_{\max}(\{C\})$ is finite if and only if C is a convex polygon.*

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Suppose, for contradiction, that one of the bodies in $\mathcal{F}$, say C_1, is not a polygon. This means that C_1 has infinitely many exposed points. Let k be an arbitrary large positive integer. As in the proof of Proposition 2, we start with a convex polygon P with k vertices circumscribed about C_1. Let $w_1, w_2, \dots, w_k$ denote the vertices of P in the clockwise order and let $x_1, x_2, \dots, x_k$ be the (unique) points on the boundary of C_1 at which the sides $[w_1, w_2], [w_2, w_3], \dots, [w_k, w_1]$ touch it. Furthermore, let $\gamma_1, \gamma_2, \dots, \gamma_k$ denote the arcs on the boundary of C_1 with endpoints x_k and x_1, x_1 and x_2, $\dots$, x_{k-1} and x_k, respectively. (See Figure 2 for the case $n = 4$.) For $1 \leqslant i \leqslant k$, choose a convex polygonal line L_i with the vertices $w_1^{(i)}, w_2^{(i)}, \dots, w_{j_i}^{(i)}$ in the triangular region bounded by the segments $[x_{i-1}, w_i]$, $[w_i, x_i]$ and the arc γ_i, satisfying the conditions $w_1^{(i)} = x_{i-1}$, $w_2^{(i)} \in]x_{i-1}, w_i[$, $w_{j_i-1}^{(i)} \in]w_i, x_i[$, $w_{j_i}^{(i)} = x_i$ and $L_i \cap \mathcal{F} = \{x_{i-1}, x_i\}$ (setting $x_0 = x_k$). Obviously, such polygonal lines $L_1, L_2, \dots, L_k$ exist since the bodies $C_1, C_2, \dots, C_n$ are disjoint. In this way we obtain a polygon $\overline{P}$ bounded by $L_1, L_2, \dots, L_k$ which contains C_1 and touches it only at the points $x_1, x_2, \dots, x_k$ (see Figure 2).

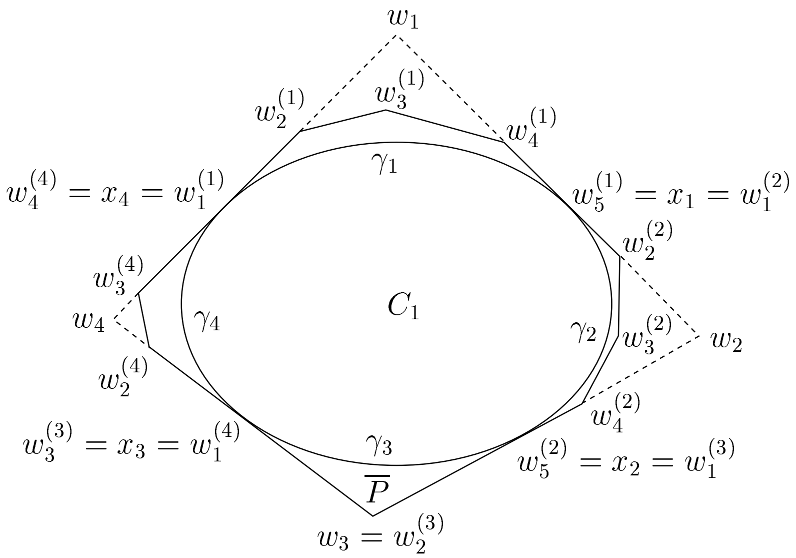


Figure 2

Since C_1 has infinitely many exposed points, we can choose the points $x_1, x_2, \dots, x_k$ so that none of them lies on the common supporting lines of the pairs of the bodies $C_1, C_2, \dots, C_n$. We may also assume that no two of the above common supporting lines intersect each other on the boundary of $\overline{P}$. These technical requirements will be used in the remaining part of the proof.

Our strategy is to construct a finite (or infinite in certain cases) illumination system containing a primitive subsystem of cardinality at least k. First consider the set S of those points in the plane which are illuminated by $\overline{P}$. We say that the points of the complement of S are in the shadow. It is easy to see that one can choose a finite pointset Q lying entirely in the shadow which can illuminate all boundary points of the bodies $C_2, \dots, C_n$ also being in the shadow except for (at most) a finite number of line segments on the boundary of $C_2, \dots, C_n$. Note that Q illuminates no point of $\overline{P}$. Now consider the mentioned finitely many line segments. These segments lie on the boundary of the shadow (in fact, they are on a common supporting line of some two of $\overline{P}, C_2, \dots, C_n$). Choosing illuminating points for the segments sufficiently close to their midpoints, the finite set Q' of these points illuminates all the segments, but does not illuminate entirely any of the arcs $\gamma_1, \gamma_2, \dots, \gamma_k$.

We turn to the illumination of those points of the bodies which lie in S. Let u be any such point. Without loss of generality we may assume that u is on the boundary of C_2. Since $u \in S$, there exists a point $v \in \overline{P}$ which illuminates u. Assign to u an arc consisting of those boundary points u' of C_2 for which there exists a point $v' \in \overline{P}$ with the property that we can move continuously v' to v and u' to u so that at every intermediate stage (including the ends) the corresponding point v^* illuminates u^*. In this way we obtain a connected arc on the boundary of C_2, but we do not know whether this arc is open, half-open or closed.

Carrying out this process for all points of S we obtain finitely many (not necessarily disjoint) arcs on the boundary of $C_2, \dots, C_n$. It follows from a simple compactness argument that any closed subarc of these arcs can be illuminated by a finite pointset of $\overline{P} \setminus \{x_1, \dots, x_k\}$, so we only deal with the illumination of the open and half-open arcs, more precisely, with the illumination of the open ends of these arcs. Let α be an open arc and let z be one of its endpoints (the case of a half-open arc is identical to the case of an open arc). If there is a point on $\overline{P}$ which illuminates a neighbourhood of z (excluding the point z, of course) of α, then there is no difficulty in the illumination of this end of α. But generally we cannot find a point on $\overline{P}$ with this property. Then z lies on a common separating supporting line of two of the bodies $C_2, \dots, C_n$.

Now, except for some special cases, e.g. z is in the interior of another arc or some point in $Q \cup Q'$ illuminates a neighbourhood of z, there does not exist a finite illumination system on $\overline{P}$ for any neighbourhood of z. In this case, as it will be detailed in the proof of Theorem 2, we can select infinitely many points on a suitable small part of $\overline{P} \setminus \{x_1, x_2, \dots, x_k\}$ following each other according to some orientation of $\overline{P}$, forming a convergent sequence, and illuminating a neighbourhood of z on α in a primitive way. Furthermore, we can select this sequence of points so that there is no vertex of $\overline{P}$ or any previously selected i lluminating point for $\mathcal{F}$ between the first point and the accumulation point of the sequence.

We construct these sequences for every open end of the open and half-open arcs defined above. Since no two of the common supporting lines of the pairs of the bodies $C_2, \dots, C_n$ intersect each other on the boundary of $\overline{P}$, the sequences do not overlap (i.e. they can be separated by points) on $\overline{P}$. Let $R_1, R_2, \dots, R_t$ denote these sequences. Observe that $R_1 \cup R_2 \cup \dots \cup R_t$ is a primitive illumination system.

To finish the proof consider the union V' of $Q \cup Q'$, of the finite illumination systems of suitable large closed subarcs of the arcs defined above and of the vertex set of $\overline{P}$, and of $R_1 \cup R_2 \cup \dots \cup R_t$. Obviously, V' is an illumination system for $\mathcal{F}$ and it is easy to see that no point of $R_1 \cup R_2 \cup \dots \cup R_t$ can be omitted from V'. So we can select from V' a primitive illumination system V for $\mathcal{F}$. But V must contain at least k points, since the arcs $\gamma_1, \gamma_2, \dots, \gamma_k$ require at least k different illuminating points, a contradiction. $\square$

Proof of Theorem 2. To show that the necessary condition for the finiteness of $p_{\max}(\mathcal{F})$ in Theorem 1 is not sufficient, consider in the Cartesian plane the triangles

$$\begin{aligned} T_1 &: a_1 = (1; 0),\ b_1 = (4; 0),\ c_1 = (1; 1), \\ T_2 &: a_2 = (-1; 0),\ b_2 = (-4; 0),\ c_2 = (-1; 1), \\ T_3 &: a_3 = (0; -2),\ b_3 = (-3; -1),\ c_3 = (3; -1), \end{aligned}$$

as shown in Figure 3.

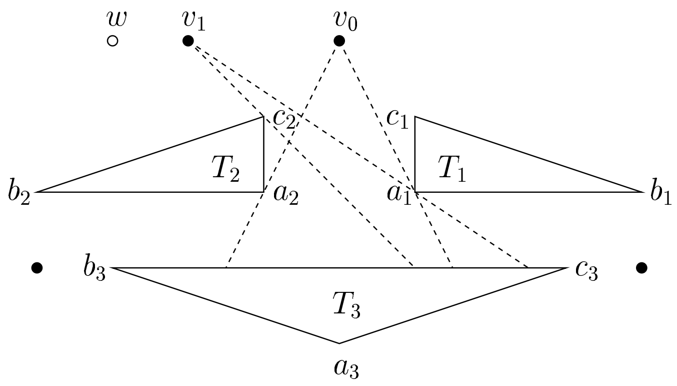


Figure 3

We shall build an infinite primitive illumination system for these triangles. First pick the point $v_0 = (0; 2)$. It illuminates the sides $[a_1, c_1]$, $[b_1, c_1]$, $[a_2, c_2]$, $[b_2, c_2]$ and the open interval $]u_0, z_0[$ on $[b_3, c_3]$, where $u_0 = (-1.5; -1)$ and $z_0 = (1.5; -1)$. Now, one can easily select recursively a sequence of points $v_1, v_2, \dots$ in $]w, v_0[$, where $w = (-3; 2)$, which satisfies the following conditions:

(1) $\lim_{i\to\infty} v_i = w$,

(2) the open intervals $]u_i, z_i[\subseteq [b_3, c_3]$ illuminated by the points v_i, $i = 0, 1, 2, \dots$, together cover the open interval $]u_0, c_3[$, but no proper subfamily of the intervals $]u_i, z_i[$ covers $]u_0, c_3[$.

If we reflect the points $v_0, v_1, v_2, \dots$ with respect to the y-axis, the obtained point set with the original one (omitting the point v_0 if necessary) forms a primitive illumination system for $]b_3, c_3[$. Adding two further illuminating points $(-4; -1)$ and $(4; -1)$, the result is an infinite primitive illumination system for $\{T_1, T_2, T_3\}$. $\square$

4. Proof of Theorem 3 and Theorem 4

Proof of Theorem 3. Let us consider one of the polygons of $\mathcal{F}$, say C_1, and one of its side, say $[w_1, w_2]$. If v is an illuminating point lying outside $\mathcal{F}$, then v can illuminate finitely many disjoint intervals on $[w_1, w_2]$. For one such interval we have the following possibilities.

(1) It is the single point w_1.

(2) It is the single point w_2.

(3) It is the whole side $[w_1, w_2]$.

(4) It is an interval $[w_1, u[$, where $u \in]w_1, w_2]$.

(5) It is an interval $]u, w_2]$, where $u \in [w_1, w_2[$.

(6) It is an interval $]w_1, u[$, where $u \in]w_1, w_2]$.

(7) It is an interval $]u, w_2[$, where $u \in [w_1, w_2[$.

(8) It is an interval $]u_1, u_2[$, where $u_1, u_2 \in]w_1, w_2[$.

The condition that $\mathcal{F}$ does not contain a critical triple immediately implies that there exists a universal constant $\varepsilon > 0$ (depending on just $\mathcal{F}$) such that the length of any interval of types (6)-(7)-(8) is at least ε. Using this observation we shall prove that $[w_1, w_2]$ requires at most k illuminating points from any primitive illumination system V for $\mathcal{F}$, where k also is a universal constant (depending on $\mathcal{F}$). From this argument Theorem 3 follows immediately.

Apart from some trivial or mixed situation we have only two essentially different cases.

Case 1. w_1 is covered by an interval of type (4) and w_2 is covered by an interval of type (5). Consider all intervals on $[w_1, w_2]$ illuminated by the points of V. A standard compactness argument shows that there exists a subfamily of these intervals consisting of exactly one interval of type (4), exactly one interval of type (5) and finitely many intervals of types (6)-(7)-(8) which also covers $[w_1, w_2]$. Observe that if a point of $[w_1, w_2]$ belongs to three intervals of this subfamily, we can omit one of them. So, since the length of the intervals of types (6)-(7)-(8) is at least ε, the assertion holds.

Case 2. w_1 is not covered by any interval of type (4) and w_2 is not covered by any interval of type (5). If the covering system contains an interval of type (6) and an interval of type (7), then the same argument works as in Case 1. Suppose, for contradiction, that the covering system does not contain intervals of type (6) or of type (7), say of type (6). Now, we have an infinite family $\{]u_1^{(i)}, u_2^{(i)}[\}_{i\in I}$ (I is an appropriate index set) of intervals of type (8) on $[w_1, w_2]$ with the property that the points $u_1^{(i)}$ accumulate to w_1. Since the length of these intervals is at least ε we can omit infinitely many of them keeping the covering property. Moreover, we can omit infinitely many intervals so that there are infinitely many illuminating points in V which illuminate only these omitted intervals on $[w_1, w_2]$. Since the illumination system is primitive this infinite subfamily $V' \subseteq V$ has to occur at the illumination of the other sides of the polygons of $\mathcal{F}$. Now, there is at least one side which is illuminated (not necessarily completely) by infinitely many points of V'. Repeating the same arguments as above for this side, we again find infinitely many unnecessary illuminating points in V'. Applying this argument at most finitely many times, finally we obtain a contradiction. $\square$

Proof of Theorem 4. The counterexample consists of one quadrangle Q and three triangles T_1, T_2, T_3.

$$\begin{aligned}
Q &: a = (1.5; 2),\ b = (6.5; 3),\ c = (6.5; -3),\ d = (1.5; -2),\\
T_1 &: a_1 = (0; 0),\ b_1 = (-0.5, 1),\ c_1 = (-0.5; -1),\\
T_2 &: a_2 = (0; 3),\ b_2 = (5; 4.5),\ c_2 = (5; 3),\\
T_3 &: a_3 = (0; -3),\ b_3 = (5; -4.5),\ c_3 = (5; -3),
\end{aligned}$$

as shown in Figure 4.

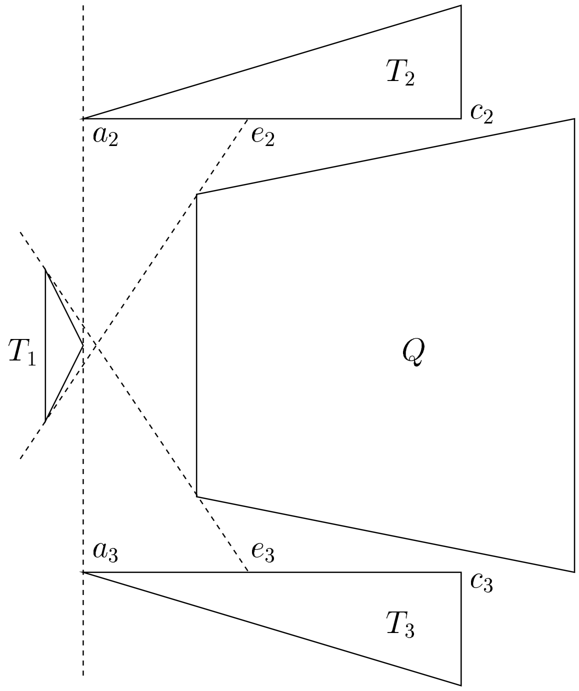


Figure 4

Analyzing the proof of Theorem 3, it is easy to see that we have to deal only with the illumination of the line segments $[a_2, e_2]$ and $[a_3, e_3]$ on the sides $[a_2, c_2]$ and $[a_3, c_3]$ of T_2 and T_3, respectively, where $e_2 = (\frac{13}{6}; 3)$ and $e_3 = (\frac{13}{6}; -3)$. Now observe that if a point illuminates any point of the open line segment $]e_2, c_2[$ ($]e_3, c_3[$ respectively), then this point also illuminates the whole line segment $[a_2, e_2]$ ($[a_3, e_3]$ respectively). From this the finiteness of $p_{\max}(\{Q, T_1, T_2, T_3\})$ follows immediately. □

5. Proof of Theorem 5.

We will show that the configuration conditions in Theorem 5 imply the existence of an infinite primitive illumination system for $\{P_1, P_2, P_3\}$.

Proof of Theorem 5. We may assume that the line ℓ is vertical, the polygons P_1 and P_3 are above and below in the left-hand side of ℓ and the polygon P_2 is in the right-hand side of ℓ. Let ℓ_1, ℓ_2, ℓ_3 be the other vertical supporting lines of P_1, P_2, P_3, respectively. Let $a_1, a_2, \dots, a_r$ denote the vertices of P_1 in the clockwise order. Suppose that the lines ℓ and ℓ_1 support P_1 at the sides $[a_1, a_r]$ and $[a_k, a_{k+1}]$, respectively, where the vertices a_1 and a_r or the vertices a_k and a_{k+1} may coincide. Similarly, let $c_1, c_2, \dots, c_t$ be the vertices of P_3 in the counterclockwise order. Let the lines ℓ and ℓ_3 support P_3 at the sides $[c_1, c_t]$ and $[c_m, c_{m+1}]$, respectively, where $c_1 = c_t$ or $c_m = c_{m+1}$ may happen. Finally, let $b_1, b_2, \ldots, b_s$ denote the vertices of P_2 in the clockwise order. Suppose that the lines ℓ and ℓ_2 support P_2 at the sides $[b_1, b_s]$ and $[b_l, b_{l+1}]$, respectively, where the vertices b_1 and b_s or the vertices b_l and b_{l+1} are not necessarily distinct (see Figure 5).

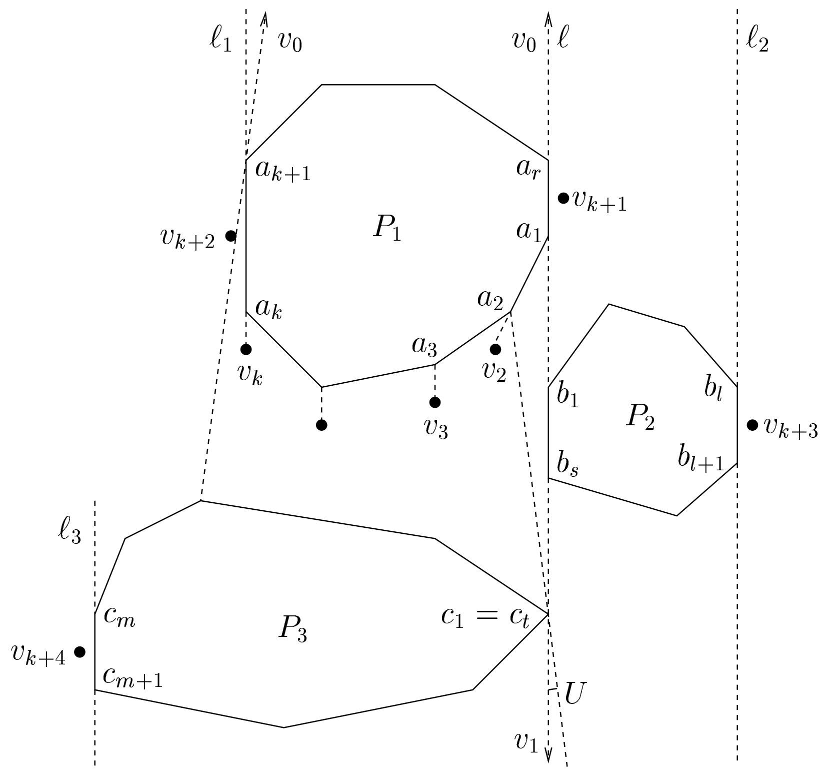


Figure 5

Without loss of generality we may assume that the distance of a_2 from ℓ is not greater than the distance of c_2 from ℓ. As in the proof of Theorem 2, we can build an infinite primitive illumination system U for the side $]a_1, a_2[$ on a horizontal (open) line segment with one endpoint on ℓ and the other endpoint on the line determined by the points a_2 and c_1 lying sufficiently close to the vertex c_1. We proceed by giving a finite illumination system V which together with U illuminates $\{P_1, P_2, P_3\}$ but no point of V illuminates the points of $]a_1, a_2[$. Let v_2 be a point on the line determined by a_1 and a_2 sufficiently close to a_2. This point illuminates the side $[b_1, b_s]$ of P_2. Next, for $3 \leqslant i \leqslant k$, choose a point v_i on the vertical line going through a_i so close to a_i that the broken line $v_2 v_3 \ldots v_k$ does not intersect the polygon

P_3. Furthermore, let v_0 and v_1 be two points above and below on the line ℓ sufficiently far from the polygons. Finally, to illuminate the sides $[a_1, a_r]$ and $[a_k, a_{k+1}]$ of P_1, $[b_l, b_{l+1}]$ of P_2 and $[c_m, c_{m+1}]$ of P_3, if necessary, we use four additional points $v_{k+1}, v_{k+2}, v_{k+3}, v_{k+4}$ sufficiently close to these sides. Let $V = \{v_0, v_1, \ldots, v_{k+4}\}$. Obviously, no point of the side $]a_1, a_2[$ of P_1 is illuminated by V and $U \cup V$ illuminates all the polygons P_1, P_2, P_3. Now, it may happen that $U \cup V$ is not a primitive system but it is easy to see that by omitting some points of V we can obtain a primitive one for our polygons. This system contains infinitely many illuminating points since it contains U. $\Box$

We note that the additional separation condition in Theorem 5 cannot be dropped. Indeed, one can construct a critical triple $\mathcal{F}$ consisting of three disjoint triangles for which $p_{\max}(\mathcal{F})$ is finite (see Figure 6).

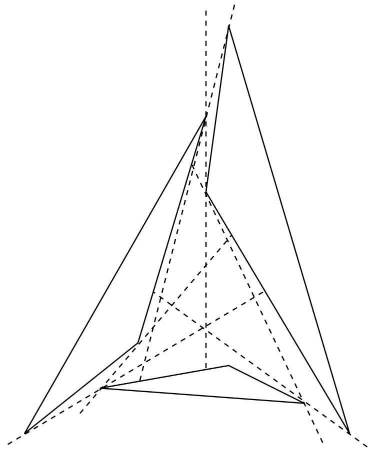


Figure 6

So, it is not a surprise that a sufficient and necessary condition for the finiteness of $p_{\max}(\mathcal{F})$ for three convex polygons is very complicated.

Let P_1, P_2, P_3 be three disjoint convex polygons forming a critical triple in this order. Let ℓ_1 and ℓ_2 be the two common separating supporting lines of P_1 and P_2 where ℓ_1 is a critical line. Let b_1 and c^* denote the points at which ℓ_1 enters P_2 and P_3, respectively (we assume that the line ℓ_1 is oriented and meets P_1, P_2, P_3 in this order). Finally, let $[c_1, c_2]$ be the side of P_3 containing c^* where c_2 lies in the open halfplane bounded by ℓ_1 and not containing P_2. Now, one can show in exactly the same way as in the proof of Theorem 5 that $p_{\max}(\{P_1, P_2, P_3\}) = \infty$ if one of the following cases holds.

Case 1. (Figure 7(a)).
(1) $c^* = c_1$,
(2) $P_2 \cap \ell_1$ is a single point,
(3) $\ell_2 \cap]c_1, c_2[= \emptyset$.

Case 2. (Figure 7(b)).
(1) $c^* = c_1$,
(2) $P_2 \cap \ell_1$ is a single point,
(3) $\ell_2 \cap]c_1, c_2[\neq \emptyset$,
(4) the line c_1c_2 is disjoint from P_1.

Case 3. (Figure 7(c)).
(1) $c^* = c_1$,
(2) $P_2 \cap \ell_1$ is a side of P_2,
(3) $\ell_2 \cap [c_1, c_2] = \emptyset$.

Case 4. (Figure 7(d)).
(1) $c^* = c_1$,
(2) $P_2 \cap \ell_1$ is a side of P_2,
(3) $\ell_2 \cap [c_1, c_2] = \{c_2\}$,
(4) $[b_1, c_2] \cap P_1 = \emptyset$.

Case 5. (Figure 7(e)).
(1) $c^* = c_1$,
(2) $P_2 \cap \ell_1$ is a side of P_2,
(3) $\ell_2 \cap [c_1, c_2] = \{c_2\}$,
(4) $[b_1, c_2] \subseteq \ell_2$,
(5) ℓ_2 is a support line of P_3.

Case 6. (Figure 7(f)).
(1) $c^* = c_1$,
(2) $P_2 \cap \ell_1$ is a side of P_2,
(3) $\ell_2 \cap]c_1, c_2[\neq \emptyset$,
(4) $[b_1, c_2] \cap P_1 = \emptyset$,
(5) the line c_1c_2 is disjoint from P_1.

Case 7. (Figure 7(g)).
(1) $c^* = c_1$,
(2) $P_2 \cap \ell_1$ is a side of P_2,
(3) $\ell_2 \cap]c_1, c_2[\neq \emptyset$,
(4) the line b_1c_2 supports P_1 and P_3,
(5) the line c_1c_2 is disjoint from P_1.

Case 8. (Figure 7(h)).
(1) $c^* \in]c_1, c_2[$,
(2) the line c_1c_2 is disjoint from P_2,
(3) $\ell_2 \cap [c^*, c_2] = \emptyset$.

Case 9. (Figure 7(i)).
(1) $c^* \in]c_1, c_2[$,
(2) the line c_1c_2 is disjoint from P_2,
(3) $\ell_2 \cap [c^*, c_2] = \{c_2\}$,
(4) $[b_1, c_2] \cap P_1 = \emptyset$.

Case 10. (Figure 7(j)).
(1) $c^* \in]c_1, c_2[$,
(2) the line c_1c_2 is disjoint from P_2,
(3) $\ell_2 \cap [c^*, c_2] = \{c_2\}$,
(4) $[b_1, c_2] \subseteq \ell_2$,
(5) $[b_1, c_1] \cap P_2 = \{b_1\}$.

Case 11. (Figure 7(k)).
(1) $c^* \in]c_1, c_2[$,
(2) the line c_1c_2 is disjoint from P_2,
(3) $\ell_2 \cap [c^*, c_2] = \{c_2\}$,
(4) $[b_1, c_2] \subseteq \ell_2$,
(5) $[b_1, c_2] \cap P_1$ is a single point,
(6) ℓ_2 is a support line of P_3.

Case 12. (Figure 7(l)).
(1) $c^* \in]c_1, c_2[$,
(2) the line c_1c_2 is disjoint from P_2,
(3) $\ell_2 \cap [c^*, c_2] = \{c_2\}$,
(4) $[b_1, c_2] \subseteq \ell_2$,
(5) ℓ_2 is a support line of P_3,
(6) $[b_1, c_2] \cap P_1 = [a_h, a_{h+1}]$ where b_1 is closer to a_h than to a_{h+1},
(7) $[a_h, c_1] \cap P_2 = \emptyset$.

Case 13. (Figure 7(m)).
(1) $c^* \in]c_1, c_2[$,
(2) the line c_1c_2 is disjoint from P_2,
(3) $\ell_2 \cap [c^*, c_2] = \{c_2\}$,
(4) $[b_1, c_2] \subseteq \ell_2$,
(5) ℓ_2 is a support line of P_3,
(6) $[b_1, c_2] \cap P_1 = [a_h, a_{h+1}]$ where b_1 is closer to a_h than to a_{h+1},
(7) the line a_hc_1 supports P_2 and P_3.

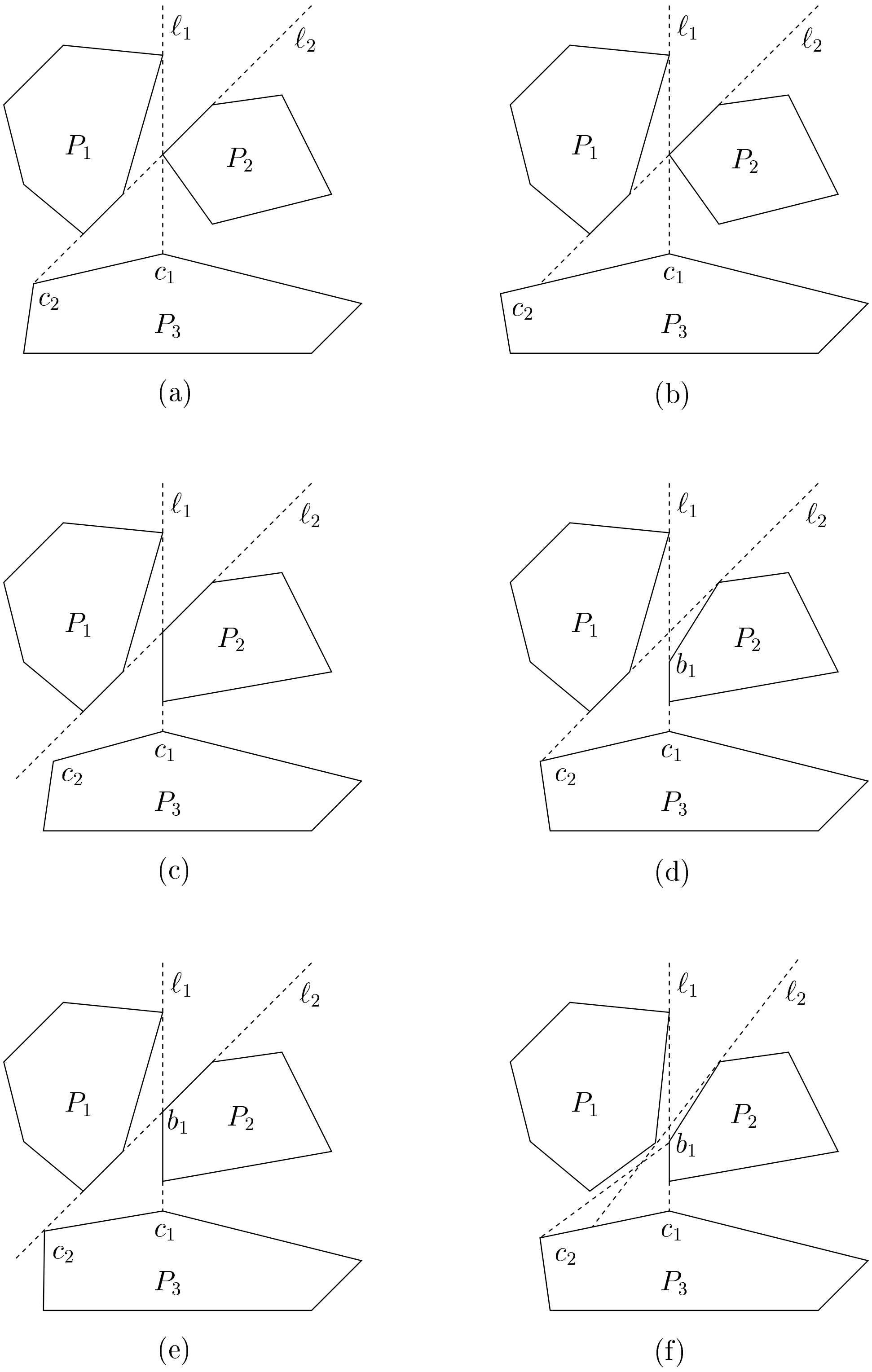
Case 14. (Figure 7(n)).
(1) $c^* \in]c_1, c_2[$,
(2) the line c_1c_2 is disjoint from P_2,
(3) $\ell_2 \cap]c^*, c_2[\neq \emptyset$,
(4) the line c_1c_2 is disjoint from P_1,
(5) $[b_1, c_1] \cap P_2 = \{b_1\}$.

Case 15. (Figure 7(o)).
(1) $c^* \in]c_1, c_2[$,
(2) the line c_1c_2 is disjoint from P_2,
(3) $\ell_2 \cap]c^*, c_2[\neq \emptyset$,
(4) the line c_1c_2 is disjoint from P_1,
(5) $[b_1, c_2] \cap P_1 = \emptyset$.

Case 16. (Figure 7(p)).
(1) $c^* \in]c_1, c_2[$,
(2) the line c_1c_2 is disjoint from P_2,
(3) $\ell_2 \cap]c^*, c_2[\neq \emptyset$,
(4) the line c_1c_2 is disjoint from P_1,
(5) $[b_1, c_2] \cap P_1$ is a single point,
(6) the line b_1c_2 supports P_3.

Case 17. (Figure 7(q)).
(1) $c^* \in]c_1, c_2[$,
(2) the line c_1c_2 is disjoint from P_2,
(3) $\ell_2 \cap]c^*, c_2[\neq \emptyset$,
(4) the line c_1c_2 is disjoint from P_1,
(5) $[b_1, c_2] \cap P_1 = [a_h, a_{h+1}]$ where b_1 is closer to a_h than to a_{h+1},
(6) the line b_1c_2 supports P_3,
(7) $[a_h, c_1] \cap P_2 = \emptyset$.

Case 18. (Figure 7(r)).
(1) $c^* \in]c_1, c_2[$,
(2) the line c_1c_2 is disjoint from P_2,
(3) $\ell_2 \cap]c^*, c_2[\neq \emptyset$,
(4) the line c_1c_2 is disjoint from P_1,
(5) $[b_1, c_2] \cap P_1 = [a_h, a_{h+1}]$ where b_1 is closer to a_h than to a_{h+1},
(6) the line b_1c_2 supports P_3,
(7) the line a_hc_1 supports P_2 and P_3.



ℓ_1
ℓ_2
P_1
P_2
P_3
c_1
c_2
(a)
ℓ_1
ℓ_2
P_1
P_2
P_3
c_1
c_2
(b)
ℓ_1
ℓ_2
P_1
P_2
P_3
c_1
c_2
(c)
ℓ_1
ℓ_2
P_1
P_2
b_1
P_3
c_1
c_2
(d)
ℓ_1
ℓ_2
P_1
b_1
P_2
P_3
c_1
c_2
(e)
ℓ_1
ℓ_2
P_1
P_2
b_1
P_3
c_1
c_2
(f)

Figure 7(a)-(f)

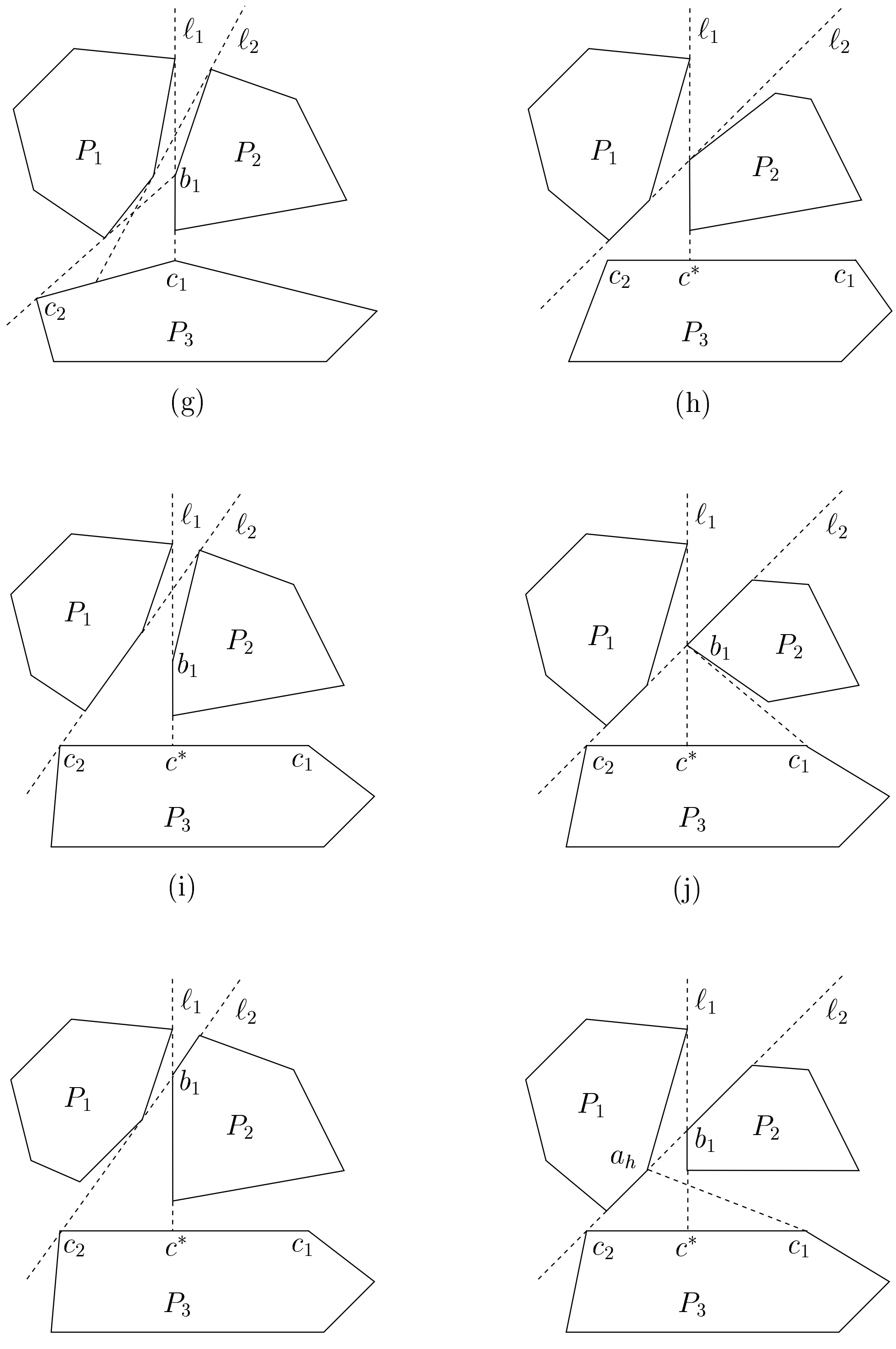

ℓ_1 ℓ_2 P_1 P_2 b_1 c_1 c_2 P_3
(g)
ℓ_1 ℓ_2 P_1 P_2 c_2 c^* c_1 P_3
(h)
ℓ_1 ℓ_2 P_1 P_2 b_1 c_2 c^* c_1 P_3
(i)
ℓ_1 ℓ_2 P_1 b_1 P_2 c_2 c^* c_1 P_3
(j)
ℓ_1 ℓ_2 b_1 P_1 P_2 c_2 c^* c_1 P_3
(k)
ℓ_1 ℓ_2 P_1 a_h b_1 P_2 c_2 c^* c_1 P_3
(l)

Figure 7(g)-(l)

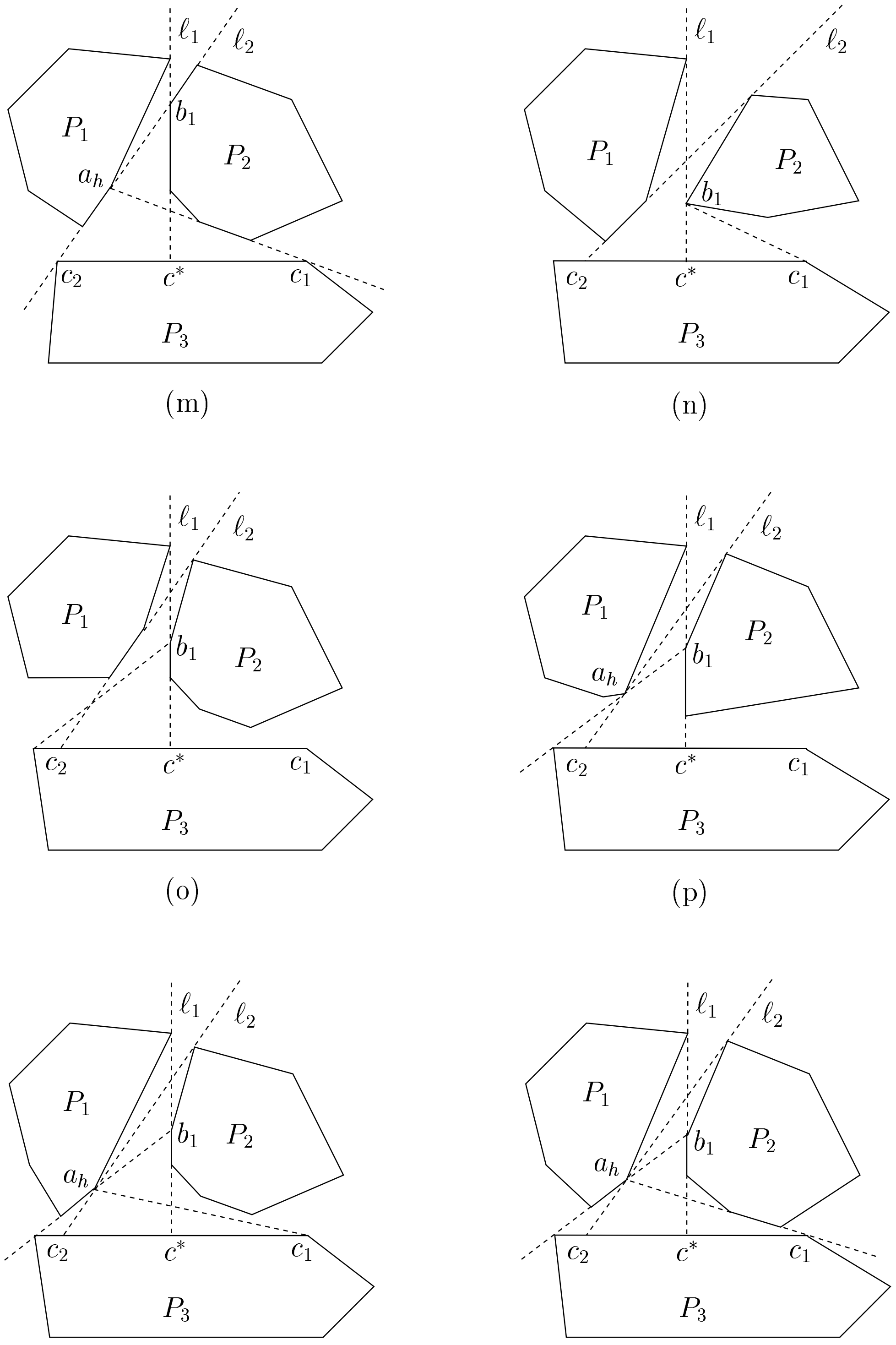


Figure 7(m)-(r)

On the other hand if $p_{\max}(\{P_1, P_2, P_3\}) = \infty$ then it is easy to see that (by the suitable choice of indices) $\{P_1, P_2, P_3\}$ must contain at least one configuration described in Cases 1-18 (note that the conditions of Theorem 5 are contained in Cases 1 and 3). The proof of this fact is very similar to the proofs of Theorems 3 and 4 and is left to the reader.

6. Proofs of Theorem 6 and Theorem 7

According to Theorems 1 and 3, the case of two convex polygons is of interest in its own right. We start with the proof of Theorem 6.

Proof of Theorem 6. In order to verify the required inequality consider a primitive illumination system V for $\{P_1, P_2\}$.

First we deal with the case when for each side of P_1 and P_2 there exists a point in V which completely illuminates it. The polygons P_1 and P_2 are disjoint, so, by the suitable choice of indices, one can find a line which contains a side f of P_1, does not intersect P_2 and separates P_1 and P_2. By our special assumption for V, there is a subset $V_1 \subseteq V$ of cardinality at most k_1 illuminating P_1. If one of the points of V_1, say v_1, illuminates the side f, then v_1 also illuminates at least one side of P_2. Hence to illuminate P_2 it is sufficient to choose at most $k_2 - 1$ further points of V. Therefore $|V| \leqslant k_1 + k_2 - 1$, since V is primitive.

Next we assume, again by the suitable choice of indices, that there is a side g of P_1 which is not illuminated by a single point of V (this situation is essentially the same as shown in Figure 8).

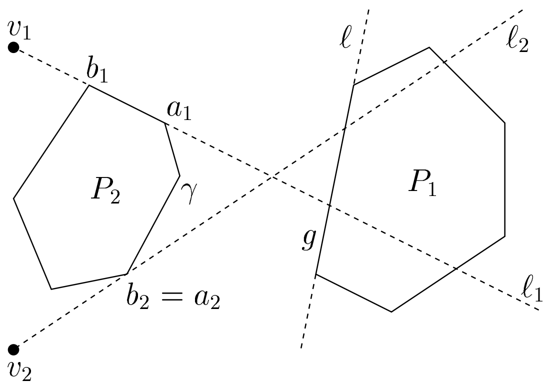


Figure 8

Now the line l containing g does not intersect P_2 and separates P_1 and P_2. For this reason g is illuminated by two points, say v_1, v_2, of V, and each of the remaining sides of P_1 can be illuminated by a single point of V. So P_1 is illuminated by a subset $V_1 \subseteq V$ with at most $k_1 + 1$ points where $v_1, v_2 \in V_1$. Moreover v_1 and v_2 illuminate at least one side of P_2. For technical reasons we assume that the part of g twice-fold illuminated by v_1 and v_2 is maximal. Let l_1 and l_2 be the rays emanating from v_1 and v_2, supporting P_2 and intersecting the side g. Then the points of V lying in any of the (open) halfplanes bounded by l_1 and l_2 and not containing P_2 are in the halfplane bounded by l and containing P_1. Let the rays l_1 and l_2 support the polygon P_2 at the (not necessarily proper) line segments $[a_1, b_1]$ and $[a_2, b_2]$, respectively. Without loss of generality we may assume that $b_1 \in [a_1, v_1]$ and $b_2 \in [a_2, v_2]$. In what follows, we have to consider several subcases separately.

Case 1. Suppose that $a_1 = a_2$. If v_1 and v_2 illuminate at least two distinct sides of P_2, then one can select at most $k_2 - 2$ points in V which together with v_1 and v_2 form an illumination system for P_2. Hence P_1 and P_2 are illuminated by at most $k_1 + k_2 - 1$ points of V. Therefore $|V| \leqslant k_1 + k_2 - 1$ since V is a primitive system. We still have to deal with the case when v_1 and v_2 illuminate only one side of P_2. In this case P_2 must be a triangle with vertices a_1, b_1, b_2. Now $[a_1, b_1]$ and $[a_2, b_2]$ are illuminated by two points v_3 and v_4 of V. By the special choice of v_1 and v_2, the points v_3 and v_4 are in the halfplane bounded by l and containing P_1, therefore v_3 and v_4 illuminate at least two sides of P_1 different from g. Hence to illuminate the remaining sides of P_1 at most $k_2 - 3$ points of V are enough, so $|V| \leqslant k_2 - 3 + 4 = k_1 + k_2 - 2$ follows.

Case 2. Suppose that $a_1 \neq a_2$. Let γ denote the polygonal line on the boundary of P_2 towards g with endpoints a_1 and a_2. Obviously, the whole polygonal line γ can be seen from at most two points of V.

Case 2.1. Assume that γ is illuminated by a single point v_3 of V. Since v_3 does not illuminate g, the line l separates v_3 and P_2. Now v_3 illuminates at least one side of P_1 different from g. Hence P_1 is illuminated by a set $V_1 \subseteq V$ of at most $k_1 + 1$ points for which $v_1, v_2, v_3 \in V_1$. Furthermore, one can select $k_2 - 2$ points of V which together with v_1, v_2 and v_3 illuminate P_2. So $|V| \leqslant k_1 + k_2 - 1$.

Case 2.2. Finally assume that γ is illuminated by two points v_3 and v_4 of V. Now P_1 is again illuminated by a set $V_1 \subseteq V$ of at most $k_1 + 1$ points for which $v_1, v_2, v_3, v_4 \in V_1$, and P_2 also is illuminated by at most $k_2 - 2$ points of V together with v_1, v_2, v_3 and v_4. Therefore $|V| \leqslant k_1 + k_2 - 1$.

This completes the proof. □

Proof of Theorem 7. First we show that $p_{\max}(\{P_1, P_2\}) \geqslant 4$ for any two convex polygons. Indeed, let l_1 and l_2 be those two common supporting lines of P_1 and P_2 which do not separate them. Let us consider the angular region determined by l_1 and l_2, containing P_1 and P_2 (or the strip between l_1 and l_2 if the lines are parallel). The polygons P_1 and P_2 divide this angular region into three regions. Let us denote by R_2 the region between the polygons, and let R_1 and R_3 be the other two regions. We proceed by a case-by-case analysis.

Case 1. It may happen that one of the regions R_1 and R_3, say R_1, is not proper. In this case the apex of the above angular region must coincide with a vertex of one of the polygons, say of P_1. Choose finitely many points sufficiently close to P_1 in such a way that none of them can illuminate more than one side of P_1. They primitively illuminate P_1 and those sides of P_2 which do not lie in the region R_3 and they do not illuminate any side of P_2 lying in R_3. It is easy to see that such a point-set always exists. Since P_1 has at least three sides, our point-set consists of at least three points. Now, completing this system with one or two points from R_3 (sufficiently far from P_2) which illuminate all sides of P_2 lying in R_3 we obtain a primitive illumination system for $\{P_1, P_2\}$ having at least four points.

Case 2. Assume that all of the regions R_1, R_2, R_3 are proper. Choose one point sufficiently close to the midpoint of each side of the polygons and delete some of them to obtain a primitive illumination system V for $\{P_1, P_2\}$. Obviously, each of the regions R_1, R_2, R_3 must contain at least one point of V. If one of the lines l_1 and l_2 contains a polygon side, then this side is illuminated by a point of V different from the above points. On the other hand, if l_1 and l_2 do not contain any polygon side, then in one of the regions R_1, R_2, R_3 there are at least two points of V (this follows from the fact that P_1 and P_2 have at least three sides). So, in both cases the cardinality of V is at least four.

It remains to find two convex polygons T_1, T_2 for which $p_{\max}(\{T_1, T_2\}) = 4$. Let us consider in the Cartesian plane the triangles

$$T_1 \ : \ a_1 = (-1;0), \ b_1 = (-3;0), \ c_1 = (-2;\sqrt{3}),$$
$$T_2 \ : \ a_2 = (1;0), \ b_2 = (3;0), \ c_2 = (2;\sqrt{3}).$$

To verify the required property of $\{T_1, T_2\}$, it is convenient to divide the plane into several regions as shown in Figure 9.

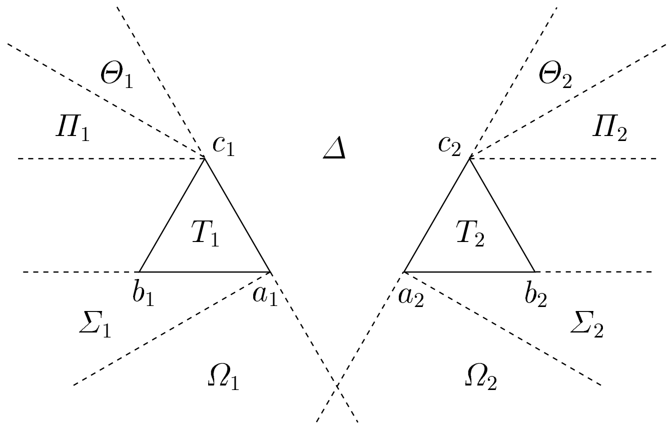


Figure 9

The proof will be done by a case-by-case analysis with respect to the location of the illuminating points for $[a_1, c_1]$ and $[a_2, c_2]$. The only difficulty is to distinguish the appropriate cases. From that point, all the cases are rather obvious. Let us consider an arbitrary primitive illumination system V for $\{T_1, T_2\}$.

CASE 1. There is a point $v \in V$ which illuminates both $[a_1, c_1]$ and $[a_2, c_2]$. Now, v must be interior to the region Δ. Since at most three further points of V illuminate the remaining sides of T_1 and T_2, we have $|V| \leqslant 4$ in this case.

CASE 2. A point $v_1 \in V$ illuminates $[a_2, c_2]$ and another point $v_2 \in V$ illuminates $[a_1, c_1]$.

CASE 2.1. $v_1 \in \Theta_1$ and $v_2 \in \Theta_2$. Now, these two points together with the point illuminating the sides $[a_1, b_1]$ and $[a_2, b_2]$ form a primitive system for $\{T_1, T_2\}$.

CASE 2.2. $v_1 \in \Theta_1$ and $v_2 \in \Omega_2$. One can see immediately that $|V| \leqslant 3$ in this case.

CASE 2.3. $v_1 \in \Omega_1$ and $v_2 \in \Theta_2$. This is exactly the same as Case 2.1.

CASE 2.4. $v_1 \in \Omega_1$ and $v_2 \in \Omega_2$. Now again $|V| \leqslant 4$.

CASE 3. Two points $v_1, v_2 \in V$ illuminate $[a_2, c_2]$ and one point $v_3 \in V$ illuminates $[a_1, c_1]$. Obviously, by a suitable choice of the indices, $v_1 \in \Pi_1$ and $v_2 \in \Sigma_1$.

CASE 3.1. $v_2 \in \Theta_2$. Then the points v_1, v_2, v_3 illuminate T_1 and T_2.

CASE 3.2. $v_3 \in \Omega_2$. Now $|V| \leqslant 4$.

Case 4. A point $v_1 \in V$ illuminates $[a_2, c_2]$ and two other points $v_2, v_3 \in V$ illuminate $[a_1, c_1]$. This is exactly the same as Case 3.

Case 5. Two points $v_1, v_2 \in V$ illuminate $[a_2, c_2]$ and two other points $v_3, v_4 \in V$ illuminate $[a_1, c_1]$. Clearly, again by a suitable choice of the indices, $v_1 \in \Pi_1$, $v_2 \in \Sigma_1$, $v_3 \in \Pi_2$ and $v_4 \in \Sigma_2$. But these four points illuminate T_1 and T_2.

This completes the proof. $\square$

7. Proof of Theorem 8 and Theorem 9

In this section by a suitable arrangement of 3 disjoint triangles we will show that $p_{\max}(3) \leqslant 6$. We also show that $p_{\max}(n) \leqslant 3n - 2$ for all $n \geqslant 4$. Unfortunately, we are not able to give information about the tightness of these bounds.

Proof of Theorem 8. Let us consider in the Cartesian plane the triangles

$$\begin{aligned} T_1 &: a_1 = (-1; 0),\ b_1 = (1; 0),\ c_1 = (0; 4\sqrt{3}), \\ T_2 &: a_2 = (-4; 0),\ b_2 = (-2; 0),\ c_2 = (-3; \sqrt{3}), \\ T_3 &: a_3 = (4; 0),\ b_3 = (2; 0),\ c_3 = (3; \sqrt{3}), \end{aligned}$$

as shown in Figure 10.

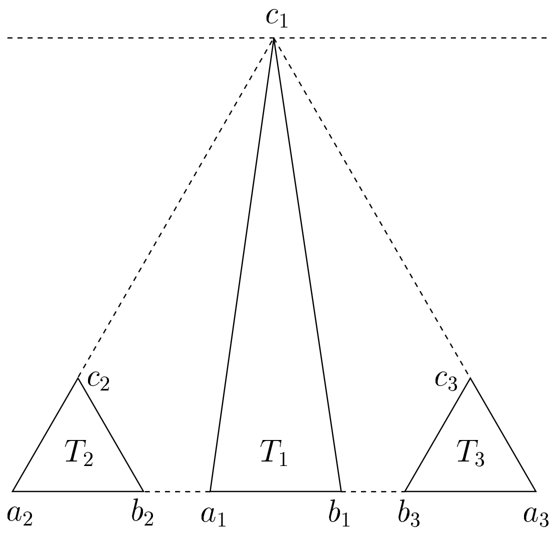


Figure 10

The proof will be done by a case-by-case analysis with respect to the number of the points illuminating the sides $[a_1, c_1]$ and $[b_1, c_1]$. Let V be an arbitrary primitive illumination system for $\{T_1, T_2, T_3\}$.

CASE 1. Each of $[a_1, c_1]$ and $[b_1, c_1]$ is illuminated by exactly two points of V. Clearly, these four points also illuminate the sides $[c_2, a_2]$, $[a_2, b_2]$, $[a_1, b_1]$, $[b_3, a_3]$, $[a_3, c_3]$. Now observe that if the illumination of $[b_2, c_2]$ ($[b_3, c_3]$) required two points of V, at least one of these points would illuminate the entire side $[b_1, c_1]$ ($[a_1, c_1]$). So, to illuminate $[b_2, c_2]$ and $[b_3, c_3]$ we need at most two additional points. Therefore $|V| \leqslant 6$.

CASE 2. $[a_1, c_1]$ is illuminated by one point of V and $[b_1, c_1]$ is illuminated by exactly two points of V. Clearly, these latter two points also illuminate the sides $[a_2, b_2]$, $[a_1, b_1]$, $[a_3, b_3]$, $[a_3, c_3]$. Furthermore, as we have already pointed out above, there is a point in V which illuminates $[b_2, c_2]$. To illuminate the side $[a_2, c_2]$ we also need at most one more point. Finally, we still have to deal with the illumination of $[b_3, c_3]$. Now the only interesting case is when $[b_3, c_3]$ is illuminated by two points. But in this case at least one of these points also illuminates $[a_1, c_1]$, so $|V| \leqslant 6$.

CASE 3. $[a_1, c_1]$ is illuminated by exactly two points of V and $[b_1, c_1]$ is illuminated by one point of V. This is exactly the same as Case 2.

CASE 4. Each of $[a_1, c_1]$ and $[b_1, c_1]$ is illuminated by one point of V. First we assume that at least one of these points, say the (highest) point which illuminates $[a_1, c_1]$, lies over the horizontal line through c_1. This point also illuminates $[a_2, c_2]$. Now, if $[b_2, c_2]$ is illuminated by exactly two points, then observe that these two points illuminate all the remaining triangle sides as well. On the other hand, if $[b_2, c_2]$ is illuminated by one point, then $|V| \leqslant 6$ again, since the point illuminating $[b_1, c_1]$ also illuminates the side $[b_3, c_3]$ or the side $[a_3, c_3]$ or all the sides $[a_2, b_2]$, $[a_1, b_1]$, $[a_3, b_3]$. Next we assume that no point of V illuminating $[a_1, c_1]$ or $[b_1, c_1]$ lies over the horizontal line through c_1. In this case observe that each side is illuminated by one point of V. Consider a point illuminating $[a_1, c_1]$. Obviously, this point also illuminates the side $[b_2, c_2]$ or the side $[a_2, c_2]$ or all the sides $[a_2, b_2]$, $[a_1, b_1]$, $[a_3, b_3]$. In each case it is clear that $|V| \leqslant 6$.

On the other hand, it is easy to see that there exist primitive illumination systems for $\{T_1, T_2, T_3\}$ consisting of exactly six points. □

Proof of Theorem 9. Let $n \geqslant 3$ and let us consider in the Cartesian plane the triangles T_i with vertices

$$a_i = \left(1; \tan\frac{\pi}{2n}\right)\begin{pmatrix} \cos\dfrac{2(i-1)\pi}{n} & \sin\dfrac{2(i-1)\pi}{n} \\ -\sin\dfrac{2(i-1)\pi}{n} & \cos\dfrac{2(i-1)\pi}{n} \end{pmatrix},$$

$$b_i = \left(1; \tan\frac{\pi}{n}\right)\begin{pmatrix} \cos\dfrac{2(i-1)\pi}{n} & \sin\dfrac{2(i-1)\pi}{n} \\ -\sin\dfrac{2(i-1)\pi}{n} & \cos\dfrac{2(i-1)\pi}{n} \end{pmatrix},$$

$$c_i = \left(1; -\tan\frac{\pi}{2n}\right)\begin{pmatrix} \cos\dfrac{2i\pi}{n} & \sin\dfrac{2i\pi}{n} \\ -\sin\dfrac{2i\pi}{n} & \cos\dfrac{2i\pi}{n} \end{pmatrix},$$

for $i = 1, 2, \ldots, n$ as shown in Figure 11. Observe that these isosceles triangles are obtained by cutting off the vertices of a regular n-gon.

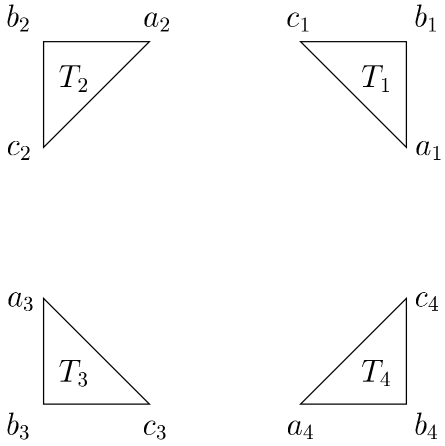


Figure 11

By Theorem 3, any primitive illumination system V for these triangles is finite. If any point of V is inside the convex polygon with vertices $a_1, c_1, a_2, c_2, \ldots, a_n, c_n$, then this point illuminates the sides $[a_1, c_1], [a_2, c_2], \ldots, [a_n, c_n]$. On the other hand, to illuminate the remaining sides of $T_1, T_2, \ldots, T_n$ at most n points of V are necessary (cf. Proposition 3). So, in this case $|V| \leqslant n+1$.

Now, we turn to the case when every point of V is outside the convex polygon with vertices $b_1, b_2, \ldots, b_n$. First we note that every side $[a_i, c_i]$ is illuminated by at most two points of V. This follows from the fact that the length of $[a_i, c_i]$ is equal to the length of $[c_i, a_{i+1}]$, $i = 1, 2, \ldots, n$. If there is a side $[a_i, b_i]$ which requires only one point of V then $|V| \leqslant 3n - 2$. Indeed, the remaining $n - 1$ sides $[a_j, b_j]$ require at most $2n - 2$ points and to illuminate the side-pairs $[b_k, c_k], [a_{k+1}, b_{k+1}]$, $k = 1, 2, \ldots, n$, at most $n - 1$ further points are already enough. On the other hand if every side $[a_i, c_i]$ is illuminated by two points of V then it is easy to see that we need at most $n - 2$ further points of V to illuminate the side–pairs $[b_k, c_k], [a_{k+1}, b_{k+1}]$, $k = 1, 2, \ldots, n$. So, also in this case $|V| \leqslant 3n - 2$ holds. □

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