Newton numbers for overlapping circular discs

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Abstract

Consider a family of closed unit circular discs in the plane. Two discs are called neighbours if they have a point in common. Let $N(d)$ denote the maximum possible number of neighbours of one disc in a family of unit circular discs where the distance between any two circle centers is at least d. Here $N(d)$ is determined for $1 < d \leq 2$.

1. Introduction

Consider a family of k closed unit circular discs in the plane. Two discs are called neighbours if they have a point in common. Let $N(k, d)$ denote the maximum possible number of neighbouring pairs among k discs where the distance between any two circle centers is at least d. It is an old result

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of Harborth that $N(k,2) = \lfloor 3k - \sqrt{12k-3} \rfloor$ (see [4]). However, it is an open problem of Radin [5] how the function $N(k,d)$ changes if d decreases. In this paper we discuss a local version of this problem. Let $N(d)$ denote the maximum possible number of neighbours of one disc in a family of unit circular discs where the distance between any two circle centers is at least d. It is clear that $N(d) = 0$ if $d > 2$, and $N(2) = 6$. Furthermore, it is also known (see [1, 2, 6]) that $N(1) = 18$. Here we determine $N(d)$ for $1 < d \leq 2$. In fact, for $6 \leq n \leq 18$ we determine the maximum positive number D_n such that $N(d) \geq n$ for $d \leq D_n$. Table 1 lists the numerical data.

n	D_n	n	D_n	n	D_n
6	2.0000 ...	11	1.1863 ...	16	1.0504 ...
7	1.7355 ...	12	1.1547 ...	17	1.0352 ...
8	1.5307 ...	13	1.1074 ...	18	1.0352 ...
9	1.3680 ...	14	1.0798 ...		
10	1.2360 ...	15	1.0677 ...		

Table 1.

2. Preliminary lemmas

Let $1 < d \leq 2$ and consider a family $\{B, B_1, \ldots, B_n\}$ of closed unit circular discs such that the distance between the centers of any two discs is at least d and $B \cap B_i \neq \emptyset$ for $1 \leq i \leq n$. Let $C, C_1, \ldots, C_n$ denote the centers of $B, B_1, \ldots, B_n$, respectively. We may assume that the halflines from C through $C_1, \ldots, C_n$, respectively, are in counterclockwise order. Throughout the paper we will use the convention that $C_{n+1} = C_1$, $C_{n+2} = C_2$, and so on.

Lemma 1. *If* $\sqrt{5} - 1 \leq d \leq 2$ *then* $\measuredangle C_i C C_{i+1} \geq \arccos(1 - d^2/8)$.

- *For* $\sqrt{5} - 1 < d \leq 2$ *equality occurs only if* $\overline{C_i C_{i+1}} = d$ *and* $\overline{CC_i} = \overline{CC_{i+1}} = 2$.
- *For* $d = \sqrt{5} - 1$ *equality occurs only if* $\overline{C_i C_{i+1}} = d$ *and either* $\overline{CC_i} = \overline{CC_{i+1}} = 2$ *or one of* $\overline{CC_i}$ *and* $\overline{CC_{i+1}}$ *is* d *while the other one is 2.*

Proof. We may assume, by symmetry, that $\overline{CC_i} \leq \overline{CC_{i+1}}$. First, if $\overline{CC_{i+1}} < 2$ then translate C_{i+1} along the halfline from C through C_{i+1} until $\overline{CC_{i+1}} = 2$. In this case $\overline{C_iC_{i+1}}, \overline{CC_{i+1}}$ increase and $\measuredangle C_i C C_{i+1}$ does not change. Secondly, if $\overline{C_iC_{i+1}} > d$ then rotate C_{i+1} around C until $\overline{C_iC_{i+1}} = d$ in which case $\measuredangle C_i C C_{i+1}$ decreases. Thirdly, if $d < \overline{CC_i} < 2$ then rotate C_i around C_{i+1}

clockwise and counterclockwise until $\overline{CC_i} = d$ and $\overline{CC_i} = 2$, respectively. In at least one of the two directions $\measuredangle C_iCC_{i+1}$ decreases again. This implies that $\measuredangle C_iCC_{i+1}$ is at least the smaller one of $\psi_1(d)$ and $\psi_2(d)$ where $\psi_1(d)$ corresponds to the case $\overline{C_iC_{i+1}} = d$ and $\overline{CC_i} = \overline{CC_{i+1}} = 2$ while $\psi_2(d)$ corresponds to the case $\overline{C_iC_{i+1}} = \overline{CC_i} = d$ and $\overline{CC_{i+1}} = 2$. Moreover, equality may occur only in these two cases. For $\sqrt{5}-1 \le d \le 2$ we have

$$\begin{aligned}\cos\psi_1(d) - \cos\psi_2(d) &= 1 - \frac{d^2}{8} - \frac{1}{d} \\ &= \frac{1}{8d}(2-d)(d+1-\sqrt{5})(d+1+\sqrt{5}) \ge 0\end{aligned}$$

with equality only if $d = \sqrt{5}-1$ or $d = 2$. □

Let $r_0 = 2(\sqrt{3}-1)$. Furthermore, for $\sqrt{2}(\sqrt{3}-1) \le d < \sqrt{5}-1$ define $\varphi_I(d) = 2\arcsin\frac{d}{4}$, $\varphi_{II}(d) = \arccos\frac{1}{d}$, and $\varphi_{III}(d) = 2\arcsin\frac{d}{2r_0}$.

Lemma 2. *Let $\sqrt{2}(\sqrt{3}-1) \le d < \sqrt{5}-1$.*

- *If $\overline{CC_i}, \overline{CC_{i+1}} > r_0$ then $\measuredangle C_iCC_{i+1} \ge \varphi_I(d)$ with equality only if $\overline{C_iC_{i+1}} = d$ and $\overline{CC_i} = \overline{CC_{i+1}} = 2$.*
- *If $\overline{CC_i} \le r_0$ and $\overline{CC_{i+1}} > r_0$ then $\measuredangle C_iCC_{i+1} \ge \varphi_{II}(d)$ with equality only if $\overline{C_iC_{i+1}} = \overline{CC_i} = d$ and $\overline{CC_{i+1}} = 2$.*
- *If $\overline{CC_i}, \overline{CC_{i+1}} \le r_0$ then $\measuredangle C_iCC_{i+1} \ge \varphi_{III}(d)$ with equality only if $\overline{C_iC_{i+1}} = d$ and $\overline{CC_i} = \overline{CC_{i+1}} = r_0$.*

Proof. In the proof of Lemma 1 we have shown that $\measuredangle C_iCC_{i+1}$ is minimal only if $\overline{CC_{i+1}}$ is maximal, $\overline{CC_i}$ is maximal or minimal, and $\overline{C_iC_{i+1}} = d$. From this the assertions follow by straightforward calculations which are left to the reader. □

Also straightforward calculations show

Lemma 3. *For $\sqrt{2}(\sqrt{3}-1) \le d < \sqrt{5}-1$ we have $\varphi_{II}(d) < \varphi_I(d) < \varphi_{III}(d)$. Furthermore, $\varphi_I(d) \ge \pi/6$ and $\varphi_{II}(d) \ge \pi/12$ with equality in both cases only if $d = \sqrt{2}(\sqrt{3}-1)$.*

Lemma 4. *For $\sqrt{2}(\sqrt{3}-1) \le d < 2/\sqrt{3}$ we have $\measuredangle C_iCC_{i+2} \ge 2\,\varphi_{II}(d)$ with equality only if $\overline{CC_i} = \overline{CC_{i+2}} = 2$ and $\overline{CC_{i+1}} = \overline{C_iC_{i+1}} = \overline{C_{i+1}C_{i+2}} = d$. In addition, if $\overline{CC_{i+1}} \ge r_0$ and $\overline{CC_i}, \overline{CC_{i+2}} \le r_0$ then $\measuredangle C_iCC_{i+2} \ge \varphi_{II}(d) + \pi/6$ with equality only if $\overline{CC_{i+1}} = 2$, $\overline{C_iC_{i+1}} = \overline{C_{i+1}C_{i+2}} = \overline{C_iC_{i+2}} = d$, and either $\overline{CC_i} = d$ or $\overline{CC_{i+2}} = d$.*

Proof. From Lemma 2 and Lemma 3 with the observation that if $\overline{CC_i} = \overline{CC_{i+2}} = 2$ and $\overline{CC_{i+1}} = \overline{C_iC_{i+1}} = \overline{C_{i+1}C_{i+2}} = d$ then $\overline{C_iC_{i+2}} \geq d$ with equality only if $d = \sqrt{2}(\sqrt{3}-1)$ the first part of Lemma 4 follows.

We turn to the second part of Lemma 4. Assume that $\measuredangle C_iCC_{i+2} \leq \varphi_{II}(d) + \pi/6$ otherwise there is nothing to prove. Translate C_{i+1} along the halfline from C through C_{i+1} until $\overline{CC_{i+1}} = 2$. This does not violate the conditions on the points. We note that C_i and C_{i+2} lie in the curvilinear triangles bounded by the circles of radius d with centers C and C_{i+1}, and by the halflines from C at angle $\pi/6$ clockwise and counterclockwise, respectively, to the halfline from C through C_{i+1}, since $\measuredangle C_iCC_{i+1}, \measuredangle C_{i+1}CC_{i+2} \geq \varphi_{II}(d)$ and $\measuredangle C_iCC_{i+2} \leq \varphi_{II}(d) + \pi/6$ (see Figure 1).

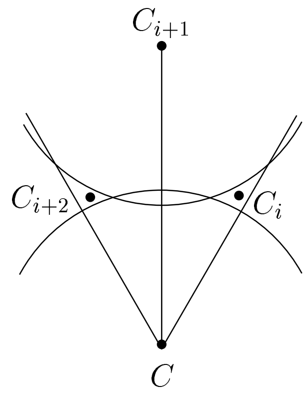

Figure 1.

Now replace C_i and C_{i+2} by the points of the halflines from C through C_i and C_{i+2}, respectively, whose distance from C_{i+1} is d. These replacements do not decrease $\overline{C_iC_{i+2}}$ since for any point P in one of the curvilinear triangles the line through P being perpendicular to CP does not intersect the other curvilinear triangle. To see this it is enough to observe that this holds for the intersection points of the circles of radius d with centers C and C_{i+1}.

We may assume, by symmetry, that $\overline{CC_i} \geq \overline{CC_{i+2}}$. Rotate C_{i+2} counterclockwise around C_{i+1} until $\overline{C_iC_{i+2}} = d$ and then rotate $\overline{C_iC_{i+2}}$ counterclockwise around C_{i+1} until $\measuredangle C_iCC_{i+1} = \pi/6$. During these rotations we have $\overline{CC_i} \leq r_0$ and $\overline{CC_{i+2}} \geq d$ since $\sqrt{2}(\sqrt{3}-1) \leq d < \frac{2}{\sqrt{3}}$. We claim that during these rotations $\measuredangle C_iCC_{i+2}$ decreases. This is trivial for the first rotation and for the second one this comes from the following observation. If we have a segment with endpoints $(-d/2, 0)$ and $(d/2, 0)$ in the coordinate system and rotate the point $(0, 2 - d\sqrt{3}/2)$ around the point $(0, -d\sqrt{3}/2)$ by an angle less than $\pi/6$ then the moving point sees the segment under a decreasing angle. This yields the second part of Lemma 4. $\square$

Lemma 5. *For $\sqrt{2}(\sqrt{3}-1) \le d < 2/\sqrt{3}$ we have $\measuredangle C_i C C_{i+5} \ge 2(\varphi_{II}(d) + \pi/6)$ if $C_i, C_{i+1}, C_{i+2}, C_{i+3}, C_{i+4}$ lie alternately inside and outside the circle around C with radius r_0. Equality occurs exactly in the following cases.*

- $\overline{CC_{i+1}} = \overline{CC_{i+3}} = 2$, $\overline{C_iC_{i+1}} = \overline{C_{i+1}C_{i+2}} = \overline{C_{i+2}C_{i+3}} = \overline{C_{i+3}C_{i+4}} = \overline{C_iC_{i+2}} = \overline{C_{i+2}C_{i+4}} = d$, *and either* $\overline{CC_{i+2}} = d$ *or* $\overline{CC_i} = \overline{CC_{i+4}} = d$.
- $\overline{CC_i} = \overline{CC_{i+2}} = \overline{CC_{i+4}} = 2$, $\overline{C_iC_{i+1}} = \overline{C_{i+1}C_{i+2}} = \overline{C_{i+2}C_{i+3}} = \overline{C_{i+3}C_{i+4}} = \overline{C_{i+1}C_{i+3}} = d$, *and either* $\overline{CC_{i+1}} = d$ *or* $\overline{CC_{i+3}} = d$.
- $\overline{CC_i} = \overline{CC_{i+4}} = 2$, $\overline{C_iC_{i+1}} = \overline{C_{i+1}C} = \overline{CC_{i+3}} = \overline{C_{i+3}C_{i+4}} = \overline{C_{i+1}C_{i+3}} = d$. *Here the position of C_{i+2} is not uniquely determined.*

Proof. If $\overline{CC_i}, \overline{CC_{i+2}}, \overline{CC_{i+4}} \le r_0$ and $\overline{CC_{i+1}}, \overline{CC_{i+3}} \ge r_0$ then Lemma 5 follows from the second part of Lemma 4. Thus assume that $\overline{CC_i}, \overline{CC_{i+2}}, \overline{CC_{i+4}} \ge r_0$ and $\overline{CC_{i+1}}, \overline{CC_{i+3}} \le r_0$. Furthermore, assume that $\measuredangle C_i C C_{i+5} \le 2(\varphi_{II}(d) + \pi/6)$ otherwise there is nothing to prove. Replace C_i, C_{i+2}, C_{i+4} by the points of the halflines from C through C_i, C_{i+2}, C_{i+4}, respectively, whose distance from C is 2. It is easy to see that these replacements do not violate the conditions on the points. Rotate C_{i+1} clockwise around C until $\overline{C_iC_{i+1}} = d$ and rotate C_{i+3} counterclockwise around C until $\overline{C_{i+3}C_{i+4}} = d$. These rotations do not decrease $\overline{C_{i+1}C_{i+3}}$. Now consider the curvilinear quadrangle bounded by the circle of radius 2 around C and by the circles of radius d around C_i, C, C_{i+4}, respectively (see Figure 2).

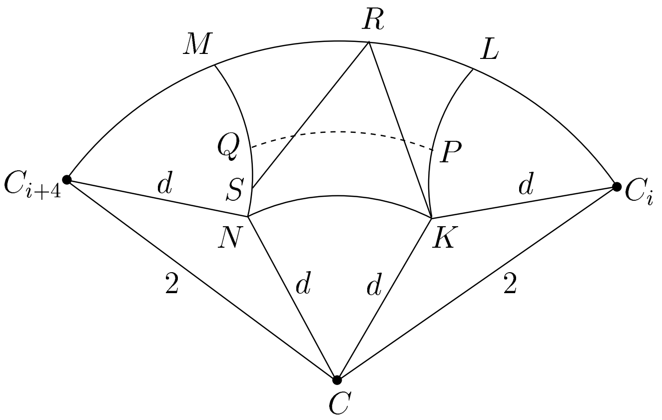

Figure 2.

Using the notations of Figure 2, the points $C_{i+1}, C_{i+2}, C_{i+3}$ lie on the arcs $\widehat{KP}$, $\widehat{LM}$, $\widehat{NQ}$, respectively, where P and Q are the points of the arcs $\widehat{KL}$ and $\widehat{NM}$, respectively, whose distance from C is r_0. We may assume, by symmetry, that $\overline{CC_{i+1}} \le \overline{CC_{i+3}}$. Let R be the (unique) point of $\widehat{LM}$ whose distance from K is d and let S be the (unique) point of $\widehat{NQ}$ whose distance from R is d. Observe that the distance of K from any point R' on the arc $\widehat{RL}$ different from R is smaller than d and if we move from K to P along $\widehat{KP}$ then the distance of the moving point from R' decreases. Therefore the point C_{i+2} lies on the arc $\widehat{RM}$. Similarly, the distance of S from any point R' on the arc $\widehat{RM}$ different from R is smaller than d and if we move from S to Q along $\widehat{SQ}$ then the distance of the moving point from R' decreases. This implies that the point C_{i+3} lies on the arc $\widehat{NS}$. Now observe that if T denotes the point on the arc $\widehat{KP}$ at distance $\overline{CS}$ from C then for any two points S' and T' on the arcs $\widehat{NS}$ and $\widehat{KT}$, respectively, $\overline{S'T'} \leq d$ with equality only if $\overline{NK} = d$ and if one of the following three situations occur, (1) $T' = K$ and $S' = N$, (2) $T' = K$ and $S' = S$, (3) $T' = T$ and $S' = N$. Thus taking into account that $\overline{C_{i+1}C_{i+3}} \geq d$ and that $\overline{NK} \leq d$ we have $\measuredangle C_iCC_{i+5} = 2(\varphi_{II}(d) + \pi/6)$ and $C_{i+1} = K$. Furthermore, for the locations of the points C_{i+2} and C_{i+3} there are only two possibilities, (1) $C_{i+2} = R$ and $C_{i+3} = S$, (2) $C_{i+3} = N$ and C_{i+2} is any point of the region bounded by the circles of radius d around K and N, and by the circle of radius 2 around C. This completes the proof. □

3. Theorems

We determine D_n for $6 \leq n \leq 18$.

Theorem 1. *Let* $d_n = 4\sin\frac{\pi}{n}$ *for* $6 \leq n \leq 10$. *Then* $D_n = d_n$ *for* $6 \leq n \leq 10$. *If* $6 \leq n \leq 9$ *then there is only one extremal configuration and if* $n = 10$ *then there are fourteen extremal configurations with* $d = D_n$.

Proof. For $6 \leq n \leq 10$, it follows from Lemma 1 that if $d \geq d_n$ then $\measuredangle C_iCC_{i+1} \geq \arccos\left(1 - 2\sin^2\frac{\pi}{n}\right) = \frac{2\pi}{n}$, $1 \leq i \leq n$, with equality only if $d = d_n$ and if the conditions for $\overline{CC_i}$, $\overline{CC_{i+1}}$, $\overline{C_iC_{i+1}}$ described in Lemma 1 hold. The extremal configurations are shown in Figures 3 and 4. □

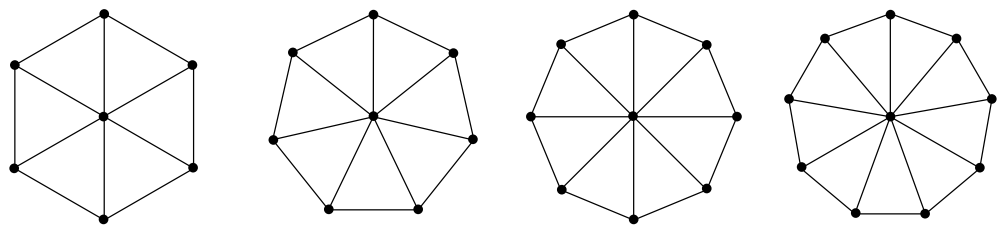

Figure 3: Extremal configurations for $n = 6, 7, 8, 9$.

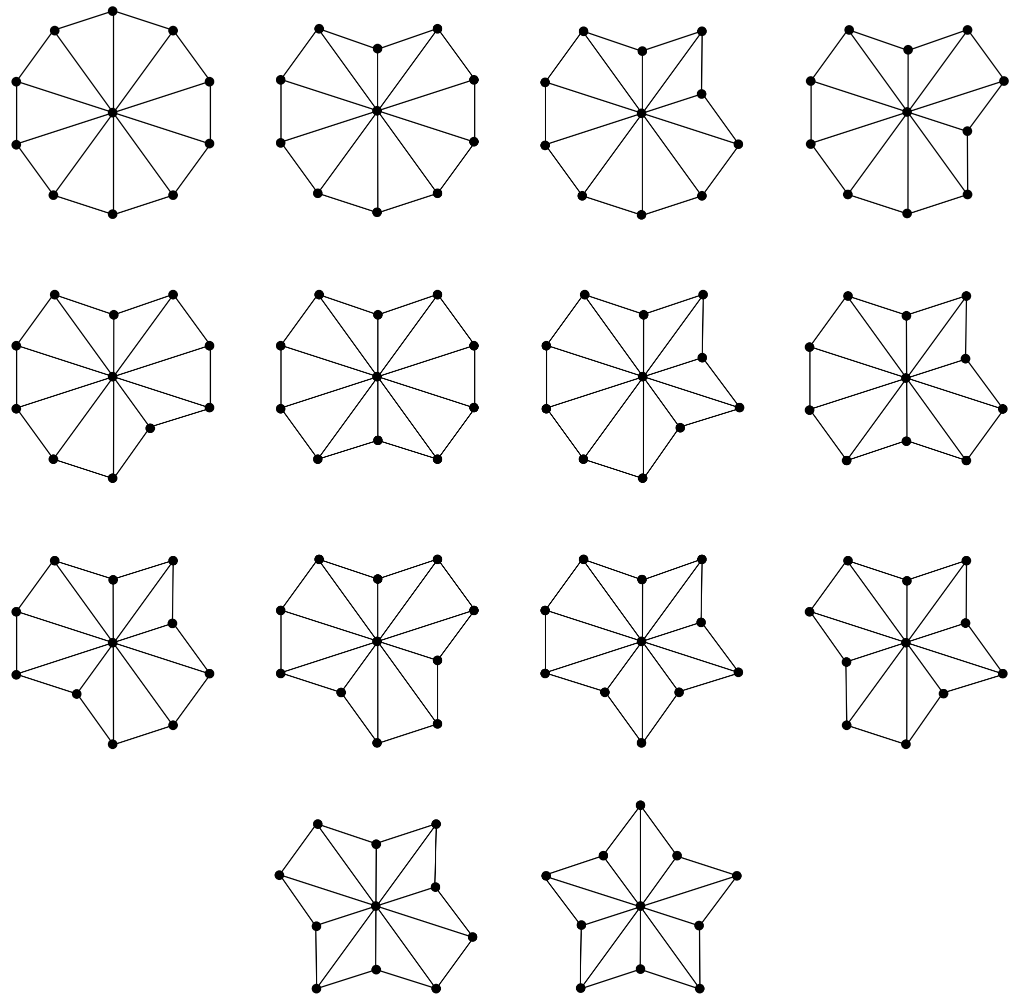

Figure 4: Extremal configurations for $n = 10$.

Theorem 2. *Let $d_{11} = 1.1863\ldots$ be the unique solution of the equation $2\arcsin\frac{d}{4} + 10\arccos\frac{1}{d} = 2\pi$ and let $d_{12} = \frac{2}{\sqrt{3}}$. Then $D_n = d_n$ for $n = 11, 12$. There is only one extremal configuration with $d = D_{11}$ and $d = D_{12}$, respectively.*

Proof. For $n = 11$, by the pigeon-hole principle at least one of the consecutive pairs $\overline{CC_i}, \overline{CC_{i+1}}$ consists of segments whose lengths are simultaneously greater or not greater than r_0. Thus

$$2\pi = \sum_{i=1}^{11} \measuredangle C_iCC_{i+1} \geq \varphi_I(d) + 10\,\varphi_{II}(d) = 2\arcsin\frac{d}{4} + 10\arccos\frac{1}{d}$$

which implies that $d \leq d_{11}$ with equality only if the discs are arranged as shown in the left part of Figure 5. For $n = 12$ we have

$$2\pi = \sum_{i=1}^{12} \measuredangle C_i C C_{i+1} \geq 12\,\varphi_{II}(d) = 12 \arccos \frac{1}{d}$$

from which $d \leq d_{12}$ with equality only if the discs are arranged as shown in the right part of Figure 5. $\square$

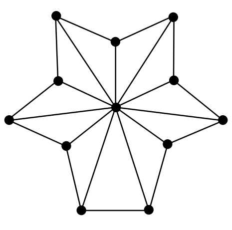

Figure 5: Extremal configurations for $n = 11, 12$.

Theorem 3. *Let $d_{13} = 1.1074\ldots$, $d_{14} = 1.0798\ldots$, $d_{15} = 1.0677\ldots$, $d_{16} = 1.0504\ldots$ be the unique solutions of the equations*

$$\begin{aligned} 6 \arcsin \frac{d}{4} + 8 \arccos \frac{1}{d} + \frac{2\pi}{6} &= 2\pi, \\ 4 \arcsin \frac{d}{4} + 8 \arccos \frac{1}{d} + \frac{4\pi}{6} &= 2\pi, \\ 10 \arcsin \frac{d}{4} + 10 \arccos \frac{1}{d} &= 2\pi, \\ 8 \arcsin \frac{d}{4} + 10 \arccos \frac{1}{d} + \frac{2\pi}{6} &= 2\pi, \end{aligned}$$

respectively, and let $d_{17} = d_{18} = \sqrt{2}(\sqrt{3}-1)$. Then $D_n = d_n$ for $13 \leq n \leq 18$. For $n = 13, 15, 16, 18$ there is only one, for $n = 14$ there are two, and for $n = 17$ there are seven essentially different extremal configurations with $d = D_n$. For $n = 17$ in one of the seven extremal configurations only 16 points are fixed and for the position of the remaining one point there are infinitely many possibilities.

Proof. Let n_1 and n_2 denote the number of points C_i for which $\overline{CC_i} \leq r_0$ and $\overline{CC_i} > r_0$, respectively. Let f_I denote the number of triangles C_iCC_{i+1} with $\overline{CC_i}, \overline{CC_{i+1}} > r_0$. Let f_{II} denote the number of triangles C_iCC_{i+1} with $\overline{CC_i} \leq r_0$ and $\overline{CC_{i+1}} > r_0$ or vice versa. Let f_{III} denote the number of triangles C_iCC_{i+1} with $\overline{CC_i}, \overline{CC_{i+1}} \le r_0$. Obviously, $n_1 + n_2 = n = f_I + f_{II} + f_{III}$, $2f_I + f_{II} = 2n_2$, $f_{II} + 2f_{III} = 2n_1$, and thus $f_I - f_{III} = n_2 - n_1$.

If $n_1 \le \left[\frac{n}{3}\right]$ then $f_I \ge n - 2\left[\frac{n}{3}\right]$ and thus

$$2\pi = \sum_{i=1}^{n} \measuredangle C_iCC_{i+1} \ge \left(n - 2\left[\frac{n}{3}\right]\right)\varphi_I(d) + 2\left[\frac{n}{3}\right]\varphi_{II}(d)$$

from which $d \le d_n$ with equality only if $n = 15, 17, 18$, $n_1 = \left[\frac{n}{3}\right]$, and $f_I = n - 2\left[\frac{n}{3}\right]$. The extremal configurations are shown in Figures 6 and 7.

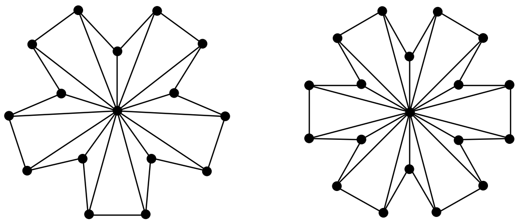

Figure 6: Extremal configurations for $n = 15, 18$.

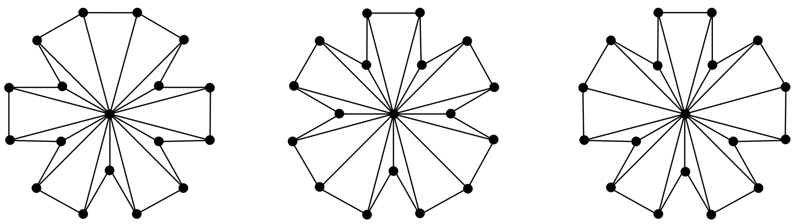

Figure 7: Extremal configurations for $n = 17$, $n_1 = 5$.

Now suppose that $n_1 \ge \left[\frac{n}{3}\right] + 1$. The number of triples (C_{i-1}, C_i, C_{i+1}), where $\overline{CC_{i-1}}, \overline{CC_{i+1}} \le r_0$ and $\overline{CC_i} > r_0$, is at least $n_2 - 2f_I$ from which, using Lemma 4,

$$\begin{aligned}\sum_{i=1}^{n} \measuredangle C_iCC_{i+1} \ge & f_I\varphi_I(d) + (f_{II} - (n_2 - 2f_I))\varphi_{II}(d) + \\ & (n_2 - 2f_I)\frac{\pi}{6} + f_{III}\varphi_{III}(d).\end{aligned} \tag{1}$$

Do not forget that $f_I + (n_2 - 2f_I) + f_{III} = n_1$ and $f_{II} - (n_2 - 2f_I) = n_2$.

If $n_1 > n - 2\left[\frac{n}{3}\right]$ then for $n = 13$ and $d \ge d_{13}$ we have

$$\sum_{i=1}^{13} \measuredangle C_iCC_{i+1} \ge \varphi_I(d) + 7\varphi_{II}(d) + \frac{5\pi}{6} > 2\pi$$

using the fact that for $n = 13$ there exist two consecutive points C_i, C_{i+1} which lie simultaneously inside or outside the circle around C with radius r_0. For $14 \le n \le 18$ we obtain

$$\sum_{i=1}^{n} \measuredangle C_i C C_{i+1} \ge \left(n + 1 - 2\left[\frac{n}{3}\right]\right)\frac{\pi}{6} + \left(2\left[\frac{n}{3}\right] - 1\right)\varphi_{II}(d) > 2\pi.$$

Thus suppose that $\left[\frac{n}{3}\right] + 1 \le n_1 \le n - 2\left[\frac{n}{3}\right]$. Since $n - 2\left[\frac{n}{3}\right] < \left[\frac{n}{3}\right] + 1$ for $n = 15, 18$ the proof is complete in these cases. Inequality (1) easily implies

$$\sum_{i=1}^{n} \measuredangle C_i C C_{i+1} \ge (n - 2n_1)\varphi_I(d) + (n - n_1)\varphi_{II}(d) + (3n_1 - n)\frac{\pi}{6}.$$

For $n = 13, 14, 16, 17$, $n_1 = n - 2\left[\frac{n}{3}\right]$ and $d \ge d_n$ the right hand side of this inequality is at least 2π with equality only if $d = d_n$. The extremal configurations are shown in Figures 8 and 9.

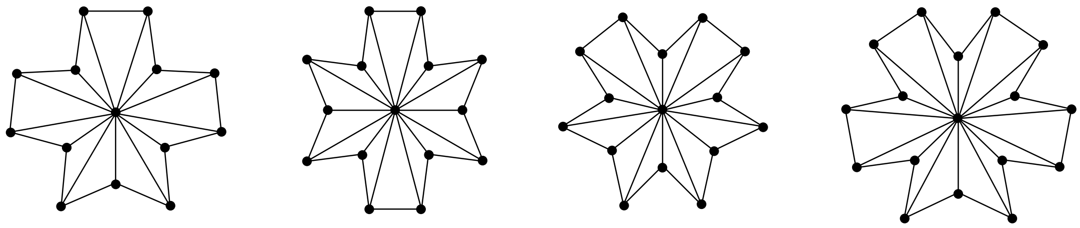

Figure 8: Extremal configurations for $n = 13, 14, 16$.

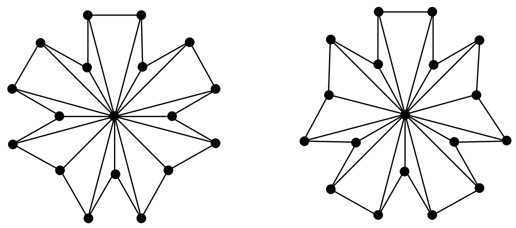

Figure 9: Extremal configurations for $n = 17$, $n_1 = 7$.

Since $n - 2\left[\frac{n}{3}\right] = \left[\frac{n}{3}\right] + 1$ for $n = 13, 16$ the proof is complete in these cases. Two cases remain, (1) $n = 14$ and $n_1 = 6$, (2) $n = 17$ and $n_1 = 6$.

For $n = 14$ and $n_1 = 6$ we have $f_I = 4$ and $f_{II} = 10$ if $f_{III} = 0$. In this case there exist five consecutive points $C_i, C_{i+1}, C_{i+2}, C_{i+3}, C_{i+4}$ which lie alternately inside and outside the circle around C with radius r_0. Hence, by Lemma 5, for $d \ge d_{14}$

$$\sum_{i=1}^{14} \measuredangle C_i C C_{i+1} \ge 4\varphi_I(d) + 8\varphi_{II}(d) + \frac{2\pi}{6} > 2\pi.$$

On the other hand, if $f_{III} \geq 1$ then $f_I \geq 5$ and thus for $d \geq d_{14}$

$$\sum_{i=1}^{14} \measuredangle C_i C C_{i+1} \geq 5\varphi_I(d) + 8\varphi_{II}(d) + \varphi_{III}(d) > 2\pi.$$

For $n = 17$ and $n_1 = 6$ we have $f_I = 5$ and $f_{II} = 12$ if $f_{III} = 0$. In this case there exist five consecutive points $C_i, C_{i+1}, C_{i+2}, C_{i+3}, C_{i+4}$ which lie alternately inside and outside the circle around C with radius r_0. Hence, by Lemma 5, for $d \geq d_{17}$

$$\sum_{i=1}^{17} \measuredangle C_i C C_{i+1} \geq 5\varphi_I(d) + 10\varphi_{II}(d) + \frac{2\pi}{6} \geq 2\pi$$

with equality only if $d = d_{17}$. The extremal configurations are shown in Figure 10. Note that in the left part of Figure 10 only 16 points are fixed and the remaining one can be anywhere in the indicated curvilinear triangle.

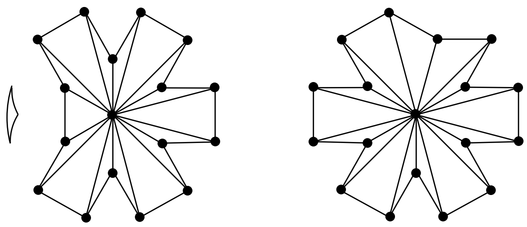

Figure 10: Extremal configurations for $n = 17$, $n_1 = 6$.

Finally, if $f_{III} \geq 1$ then $f_I \geq 6$ and thus for $d \geq d_{17}$

$$\sum_{i=1}^{17} \measuredangle C_i C C_{i+1} \geq 6\varphi_I(d) + 10\varphi_{II}(d) + \varphi_{III}(d) > 2\pi.$$

This completes the proof. □

4. Concluding remarks

In this paper we have determined the maximum possible number $N(d)$ of neighbours of one disc in a family of unit circular discs where the distance between any two circle centers is at least d for $1 < d \leq 2$. We now describe a simple bound which holds for all $0 < d \leq 2$. Consider a family $\{B, B_1, \ldots, B_n\}$ of unit circular discs such that the distance between the centers of any two discs is at least d and $B \cap B_i \neq \emptyset$ for $1 \leq i \leq n$. Let

$C, C_1, \ldots, C_n$ denote the centers of $B, B_1, \ldots, B_n$, respectively. Now the circular discs with centers $C, C_1, \ldots, C_n$ and radius $\frac{d}{2}$ form a packing in the circular disc with center C and radius $2 + \frac{d}{2}$. Then by an old result of Groemer [3]

$$n \leq \frac{\pi}{\sqrt{12}} \left(\frac{16}{d^2} + \frac{4\sqrt{3}}{d} \right)$$

which implies that

$$N(d) \leq \frac{\pi}{\sqrt{12}} \left(\frac{16}{d^2} + \frac{4\sqrt{3}}{d} \right).$$

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References

[1] Bateman, P., Erdős, P.: Geometric extrema suggested by a lemma of Besicovitch, *American Mathematical Monthly* **58** (1951), 306-314.

[2] Fejes Tóth, L.: *Lagerungen in der Ebene, auf der Kugel und in Raum*, zweite Auflage, Springer Verlag, Berlin, 1972.

[3] Groemer, H.: Über die Einlagerung von Kreisen in einen konvexen Bereich, *Mathematische Zeitschrift* **73** (1960), 285-294.

[4] Harborth, H.: Solution to problem 664A, *Elemente der Mathematik* **29** (1974), 14-15.

[5] Radin, C.: Privat communication, 1979.

[6] Reifenberg, E.R.: A problem on circles, *Mathematical Gazette* **32** (1948), 290-292.