A Simple Proof for the Jordan Measurability of Convex Sets

László Szabó*
Computer and Automation Institute
Hungarian Academy of Sciences
Lágymányosi út 11
H-1111 Budapest
Hungary

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Let B be a coordinate block in $\mathbb{E}^n$ of the form

$$B = \{(x_1, \dots, x_n) \in \mathbb{E}^n \mid a_i \leqslant x_i \leqslant b_i, \ (1 \leqslant i \leqslant n)\}$$

where $a_i < b_i$ for each i. Define the volume of B as

$$V(B) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n).$$

If $K \subseteq \mathbb{E}^n$ is a bounded set let $V^-(K) = \sup \sum_m V(B_m)$, where the supremum is taken over all packings in K by finite families $\{B_1, B_2, \dots\}$ of blocks, and let $V^+(K) = \inf \sum_m V(B_m)$, where the infimum is taken over all coverings of K by finite families $\{B_1, B_2, \dots\}$ of blocks. Let us recall that a packing in a set is an arrangement whose members are all contained in the set and have mutually disjoint interiors, and a covering of a set is an arrangement whose union contains the set. It is clear that $V^-(K) \leqslant V^+(K)$. Now, we say that the bounded set $K \subseteq \mathbb{E}^n$ is Jordan measurable if $V^-(K) = V^+(K)$, and in this case we call this common value the volume of K. For a more comprehensive account we refer the reader to the monograph [1]. The aim of this paper is to give a reasonably simple geometric proof for the following well-known

Theorem 1. *Every bounded convex set $K \subseteq \mathbb{E}^n$ is Jordan measurable.*

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Proof. Bounded convex sets of dimension less than n are clearly Jordan measurable, so we may assume that K is n-dimensional. We may also assume, without loss of generality, that K is contained in the cube

$$C = \{(x_1, \dots, x_n) \in \mathbb{E}^n \mid -\tfrac{1}{2} \leqslant x_i \leqslant \tfrac{1}{2},\ (1 \leqslant i \leqslant n)\}.$$

By subdividing each edge of C into 2^j equal parts we can partition C into congruent closed cubes, each of which has edges of length 2^{-j}. Let $\mathcal{L}_j$ denote the family of those small cubes defined above which intersect the interior of K, and let $\bar{\mathcal{L}}_j \subseteq \mathcal{L}_j$ denote the family of those small cubes which intersect the boundary of K as well ($j = 1, 2, \dots$). It is clear that $\cup(\mathcal{L}_j \setminus \bar{\mathcal{L}}_j) \subseteq K \subseteq \cup\mathcal{L}_j$ for each j.

We show that the total volume of the small cubes in $\bar{\mathcal{L}}_j$ is not greater than $n2^n2^{-j}$. Consider the 2^n directions determined by the 2^n vectors carrying the vertices of C to the origin o. Associate not greater than $n2^{j(n-1)}$ rays with each direction in the following way. For each vertex v of C consider the rays emanating from the centres of small cubes touching at least one of the n facets of C containing v, and having direction vector $\overrightarrow{vo}$. Note that the total number of these rays is not greater than $n2^n2^{j(n-1)}$ and exactly 2^n of them pass through the centre of each small cube in C.

Let C' be an arbitrary small cube in $\bar{\mathcal{L}}_j$ and consider the 2^n rays from the above family which contain the centre of C'. We claim that at least one of these rays intersects the centre of C' before it reaches (according to the natural ordering on the ray) the centre of any other small cube in $\mathcal{L}_j$. Indeed, if each of the above rays intersected the centre of some small cube in $\mathcal{L}_j$ different from C' before reaching the centre of C', then choosing one interior point of K in each of these 2^n small cubes, the convex hull of these interior points of K would contain C', so C' would not be in $\bar{\mathcal{L}}_j$. To see this we need the following simple observation.

Proposition 1. *Divide $\mathbb{E}^n$ into 2^n open connected regions by the n coordinate hyperplanes and choose one point in each of these regions. Then the convex hull of these points contains the origin.*

Proof of Proposition 1. The proof is by induction on the dimension n. For $n = 1$ the assertion is trivial. Assume that we have already proved the result for some $n \geqslant 1$, and we want to show that it also holds for $n + 1$. Let $\mathcal{P}$ denote the set of the points chosen in the regions. Let $\mathcal{P}_1$ be the set of those points of $\mathcal{P}$ whose first coordinates are negative. If Π denotes the orthogonal projection of the space onto the hyperplane of equation $x_1 = 0$, then by the induction hypothesis the convex hull of $\Pi(\mathcal{P}_1)$ contains the origin. Therefore the convex hull of $\mathcal{P}_1$ necessarily contains a point p_1 of the x_1-axis with negative first coordinate. Similarly, if $\mathcal{P}_2$ denotes the set of those points of $\mathcal{P}$ whose first coordinates are positive, then the convex hull of $\mathcal{P}_2$ contains a point p_2 of the x_1-axis with positive first coordinate. Now, the origin obviously belongs to the segment $\overline{p_1 p_2}$ and thus to the convex hull of $\mathcal{P}$ as well. $\square$

Therefore the number of small cubes in $\bar{\mathcal{L}}_j$ is not greater than $n2^n 2^{j(n-1)}$ and thus the total volume of them is at most $n2^n 2^{-j}$.

The total volumes of small cubes in $\mathcal{L}_j$ decrease while the total volumes of small cubes in $\mathcal{L}_j \setminus \bar{\mathcal{L}}_j$ increase as $j \to \infty$. Furthermore, as we have seen above, the differences between the two volumes tend to zero as $j \to \infty$. Hence the two sequences of volumes converge to the same positive number and this number is obviously equal to both $V^-(K)$ and $V^+(K)$, i.e. K is Jordan measurable. $\square$

Remark. We note that, apart from the constant $n2^n$, our estimate $n2^n 2^{-j}$ for the total volume of the small cubes in $\bar{\mathcal{L}}_j$ is the best possible for each j.

References

[1] Hadwiger, H.: *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer Verlag, Berlin, 1957.