An illumination problem for convex polyhedra

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Abstract

Consider a convex polytope P in the d-dimensional Euclidean space. We say that a vertex v of P illuminates a point $u \in E^d$ lying outside P if the line segment $\overline{uv}$ does not intersect the interior of P. Furthermore, we say that the vertices $v_1, v_2, \ldots, v_l$ of P illuminate the entire exterior of P if every point of E^d lying outside P is illuminated by at least one of the vertices $v_1, v_2, \ldots, v_l$. In this paper we consider the three dimensional situation and we prove that $\lfloor \frac{n}{2} \rfloor$ (or $\lfloor \frac{n}{2} \rfloor - 1$ if $n \equiv 2 \bmod 4$) vertices are always sufficient and sometimes necessary to illuminate the exterior of a convex polyhedron of n vertices. We give lower bounds in even dimensions as well.

1. Introduction

Let $K \subseteq E^d$ be a convex body, i.e. a compact convex set with a nonempty interior. We say that a point $q \in E^d$ exterior to the convex body K illuminates a boundary point p of K if the open ray emanating from q having direction vector $\overrightarrow{pq}$ has a nonempty intersection with the interior of K. Furthermore, we say that the points $q_1, q_2, \ldots, q_l \in E^d$ exterior to the convex body K illuminate K if every boundary point of K is illuminated by at least one of the points $q_1, q_2, \ldots, q_l$. It is a challenging open problem to determine the smallest number of points lying outside K which illuminate K for $d \geqslant 3$. For further information we refer the reader to the survey paper of Károly Bezdek [2].

In this note we deal with a related problem which was inspired by Károly Bezdek. Consider a convex polytope P in the d-dimensional Euclidean space. We say that a vertex v of P illuminates a point $u \in E^d$ lying outside P if the line segment $\overline{uv}$ does not intersect the interior of P. Furthermore, we say that the vertices $v_1, v_2, \ldots, v_l$ of P illuminate the entire exterior of P if every point of E^d lying outside P is illuminated by at least one of the vertices $v_1, v_2, \ldots, v_l$. Now, our problem can be formulated as follows. What is the smallest number of vertices of a convex polytope P of n vertices in the d-dimensional Euclidean space which illuminate the entire exterior of P.

The planar case is trivial, $\lceil \frac{n}{2} \rceil$ vertices are always sufficient and necessary to illuminate the exterior of a convex polygon of n vertices. But in higher dimensions the situation is not as straightforward. Here we solve the problem in the 3-dimensional case and give lower bounds in even dimensions as well.

Theorem 1. $\lfloor \frac{n}{2} \rfloor$ *(or* $\lfloor \frac{n}{2} \rfloor - 1$ *if* $n \equiv 2 \; mod\, 4$*) vertices are always sufficient and sometimes necessary to illuminate the exterior of a convex polyhedron of* n *vertices.*

During the preparation of this paper Godfried Toussaint pointed out to us that our problem was considered independently in [3]. However, the authors of this technical report proved Theorem 1 only for $n \equiv 3 \bmod 4$.

2. Proof of Theorem 1

First, we note that the vertices $v_1, v_2, \ldots, v_l$ illuminate the exterior of a convex polyhedron if and only if the vertices $v_1, v_2, \ldots, v_l$ cover the faces of the polyhedron, i.e. each face of the polyhedron contains at least one of these vertices on its boundary.

Now, let us see the sufficiency proof. Triangulate the faces of the polyhedron. Then four colour the vertices of this triangulated polyhedron via the Four Colour Theorem [1] and take those two colours, say black and white which together occur at most $\lfloor \frac{n}{2} \rfloor$ times (or $\lfloor \frac{n}{2} \rfloor - 1$ times if $n \equiv 2 \bmod 4$). As at each triangle of our triangulation we have three different colours the vertices coloured black and white cover all faces of the original polyhedron.

Observe that the above proof immediately leads to an $O(n^2)$ algorithm since the Four Colour Algorithm [1] runs in $O(n^2)$ time. To find linear time covering algorithm with at most $\lfloor \frac{n}{2} \rfloor$ (or $\lfloor \frac{n}{2} \rfloor - 1$ if $n \equiv 2 \bmod 4$) vertices seems to be an extremely difficult task. However, if we are satisfied with a covering system of cardinality at most $\lfloor \frac{3n}{5} \rfloor$, such linear time algorithm exists since any planar graph can be five coloured in $O(n)$ time (see [5] page 87).

To finish the proof we have to construct convex polyhedra of n vertices with the property that $\lfloor \frac{n}{2} \rfloor$ (or $\lfloor \frac{n}{2} \rfloor - 1$ if $n \equiv 2 \bmod 4$) vertices are necessary to cover their faces. Obviously, using a simple duality argument (see [4] page 46), it is enough to construct convex polyhedra of n faces with the property that $\lfloor \frac{n}{2} \rfloor$ (or $\lfloor \frac{n}{2} \rfloor - 1$ if $n \equiv 2 \bmod 4$) faces are necessary to cover their vertices. Furthermore, by a remarkable theorem of Steinitz (see [4] page 235) we can restrict ourselves to 3-connected planar graphs of n regions.

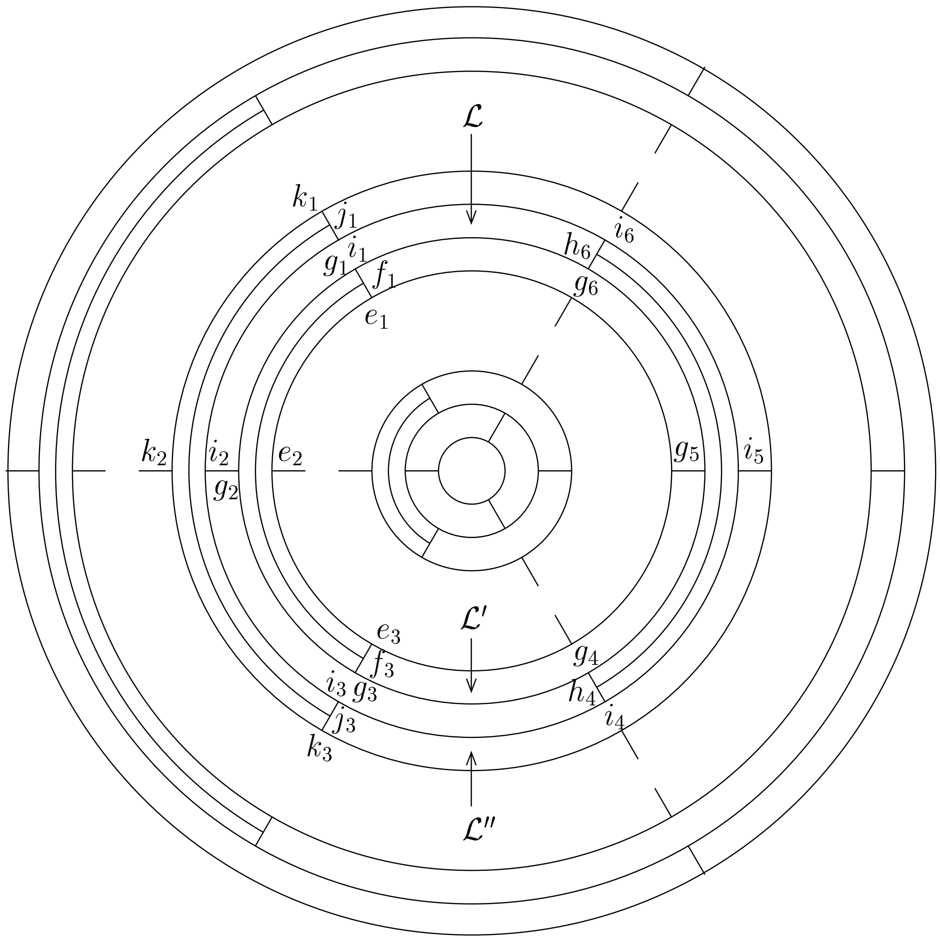

Figure 1.

For the sake of simplicity let $n = 4m$ and let us consider the 3-connected planar graph of n regions in Figure 1. Suppose that the vertices are covered by a suitable subset of the regions of this graph. Besides the "inner" and the "outer" four-four regions, our graph consists of identical layers. Consider one of these layers, say the layer $\mathcal{L}$ (the heptagons $g_2g_1g_6h_6i_6i_1i_2$ and $g_2g_3g_4h_4i_4i_3i_2$ and the pentagons $g_4g_5g_6h_6h_4$ and $i_4i_5i_6h_6h_4$ form this layer). Since the vertices h_4 and h_6 are covered, at least one region is marked in

$\mathcal{L}$. Clearly, the only interesting case is when exactly one region is marked in $\mathcal{L}$. Obviously, if this region is $g_4g_5g_6h_6h_4$ then the "next" layer $\mathcal{L}''$ must contain at least three marked regions and if this region is $i_4i_5i_6h_6h_4$ then the "previous" layer $\mathcal{L}'$ must contain at least three marked regions. We have to deal with one more case when only the regions $e_1e_2e_3f_3f_1$ and $k_1k_2k_3j_3j_1$ are marked in $\mathcal{L}'$ and $\mathcal{L}''$, respectively. But in this case all the four regions must be marked in the layer $\mathcal{L}$. These observations show that the number of the marked regions in all the layers is not smaller than half times the number of all regions in these layers. It would be still left to prove that there is no confusion about the "inner" and the "outer" parts of the graph but this can be settled similarly and the details are left to the reader. This completes the proof.

Obviously, by a slight modification of the above construction (in fact, it is enough to modify the "inner" four regions only) we can obtain the required lower bounds for the remaining cases as well.

3. Higher Dimensional Lower Bounds

In the d-dimensional Euclidean space consider the moment curve M_d defined parametrically by $x(t) = (t, t^2, \dots, t^d)$. A cyclic d-polytope $C(n,d)$ is the convex hull of $n \geqslant d+1$ points $x(t_i)$ on M_d, with $t_1 < t_2 < \cdots < t_n$. It is well-known that $C(n,d)$ is a simplicial d-polytope of n vertices (see [4] page 61).

Now, using "Gale's evenness condition" which says that a d-tuple V_d of points of $V \subseteq M_d$ determines a facet $\operatorname{conv} V_d$ of $\operatorname{conv} V = C(n,d)$ if and only if every two points of $V \setminus V_d$ are separated on M_d by an even number of points of V_d (see [4] page 62), one can prove easily that any covering of the facets of $C(n,d)$ requires at least $\lceil \frac{n}{2} \rceil - \lceil \frac{d-2}{2} \rceil$ vertices, if d is even.

First we consider the case when n is even. Then the vertices $x(t_d), x(t_{d+2}), \dots, x(t_n)$ cover the facets of $C(n,d)$ and the number of these vertices is $\lceil \frac{n}{2} \rceil - \lceil \frac{d-2}{2} \rceil$. On the other hand, if we have only $\lceil \frac{n}{2} \rceil - \lceil \frac{d}{2} \rceil$ marked vertices, then at least $\lceil \frac{d}{2} \rceil$ of the pairs $\{x(t_1), x(t_2)\}, \{x(t_3), x(t_4)\}, \dots, \{x(t_{n-1}), x(t_n)\}$ of consecutive vertices are not covered. But $\lceil \frac{d}{2} \rceil$ of these pairs represent a facet of $C(n,d)$ which is not covered.

Next we consider the case when n is odd. Now the vertices $x(t_{d-1}), x(t_{d+1}), \dots, x(t_n)$ cover the facets of $C(n,d)$. The number of these vertices is again $\lceil \frac{n}{2} \rceil - \lceil \frac{d-2}{2} \rceil$. To prove that this is the smallest number of vertices with the required property suppose that we have only $\lceil \frac{n}{2} \rceil - \lceil \frac{d}{2} \rceil$ marked vertices. Without loss of generality we may assume that the vertex $x(t_1)$ is marked (indeed, a cyclic shift of the vertices of $C(n,d)$ on the moment curve M_d does not affect the facet-structure of $C(n,d)$). Then among the pairs $\{x(t_2), x(t_3)\}, \{x(t_4), x(t_5)\}, \ldots, \{x(t_{n-1}), x(t_n)\}$ of consecutive vertices at least $\lceil \frac{d}{2} \rceil$ are not covered. But $\lceil \frac{d}{2} \rceil$ of these pairs determine again a non-covered facet of $C(n,d)$.

Thus $\lceil \frac{n}{2} \rceil - \lceil \frac{d-2}{2} \rceil$ vertices are necessary and sufficient to illuminate the exterior of $C(n,d)$, if d is even.

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