Disjoint empty convex polygons in planar point sets

Attila Gulyás
Department of Geometry
Eötvös University
Kecskeméti utca 10-12
H-1053 Budapest
Hungary

László Szabó
Computer and Automation Research Institute
Hungarian Academy of Sciences
Lágymányosi utca 11
H-1111 Budapest
Hungary

October 5, 2000

Abstract

We prove that quadruples of a set of n points in general position in the plane always induce a family of $[2n/9]$ disjoint empty convex quadrangles.

1. Introduction

Recently, the study of convex polygons has been gained a renewed interest because of their importance in computer graphics, geometric learning theory, and artificial intelligence, for instance. Surprisingly, many simple questions are unanswered in this field. Let's start with a beautiful example. We say that a set of points in the plane is in general position if no three of the points lie on a line. Decades ago, Erdős, Klein, and Szekeres posed the problem of determining the maximum number $f(k)$ of points in general position in the plane so that no k points form the vertex set of a convex polygon. Erdős and Szekeres [3] proved that

$$2^{k-2} \le f(k) \le \binom{2k-4}{k-2},$$

and conjectured that $f(k)$ is equal to the lower bound. Surprisingly, this conjecture has been verified only for $k = 3, 4, 5$. Recently, the upper bound has slightly been improved by many authors, see [2, 6, 7]. The current record, due to Tóth and Valtr [7], is

$$f(k) \le \binom{2k-5}{k-2} + 1.$$

Later, Erdős also posed the problem of determining the maximum number $g(k)$ of points in general position in the plane so that no k points form a vertex set of an empty convex polygon, i.e., a convex polygon whose interior is disjoint from the point set. It is easy to see that $g(3) = 2$ and $g(4) = 4$. Harborth [4] proved that $g(5) = 9$, and Horton [5] showed that $g(k)$ is infinite for $k \ge 7$. It is a challenging open problem to decide whether $g(6)$ is finite.

Let $g_k(n)$ denote the minimum number of empty convex k-gons induced by the k-tuples of a set of n points in general position in the plane. Bárány and Füredi [1] proved that $g_3(n) \ge n^2 - O(n \log n)$, $g_4(n) \ge \frac{1}{4}n^2 - O(n)$, and $g_5(n) \ge [n/10]$. We note that the last bound can easily be improved to $g_5(n) \ge [(n-4)/6]$. On the other hand, Valtr [8] showed that $g_3(n) \le 1.8n^2$, $g_4(n) \le 2.42n^2$, and $g_5(n) \le 1.46n^2$.

It is obvious that the k-tuples of a set of n points in general position in the plane always induce a family of $[n/(g(k)+1)]$ disjoint empty convex k-gons, and this bound is tight for $k = 3$. In this paper we consider the case $k = 4$ and we prove

Theorem 1. *The quadruples of a set of n points in general position in the plane always induce a family of $[2n/9]$ disjoint empty convex quadrangles.*

We also show that the bound $[2n/9]$ cannot be improved for $n \le 21$.

2. Proof of Theorem 1

First we prove that any set $\mathcal{P}$ of nine points in general position in the plane contains two disjoint empty convex quadrangles. Let $p_1, p_2, \ldots, p_m$ denote the vertices of the convex hull of $\mathcal{P}$ in a counterclockwise order (we will use the convention that $p_i = p_j$ if $i \equiv j \pmod{m}$). Observe that if $\triangle p_{i-1}p_ip_{i+1}$ is an empty triangle of $\mathcal{P}$ for some $1 \le i \le m$, then $\mathcal{P}$ contains two disjoint empty quadrangles. Indeed, among $\mathcal{P}\setminus\{p_{i-1}, p_i, p_{i+1}\}$ choose a point r whose distance from the line $p_{i-1}p_{i+1}$ is minimal. Now $p_{i_1}p_ip_{i+1}r$ is an empty convex quadrangle and the remaining five points of $\mathcal{P}$ also contains an empty convex quadrangle which is obviously disjoint from $p_{i-1}p_ip_{i+1}r$. Therefore, in what follows, we will assume that $\triangle p_{i-1}p_ip_{i+1}$ is not empty for $1 \le i \le m$. This immediately implies among others that $m \le 6$.

Case 1. $m = 6$. Let q_1, q_2, q_3 denote the points of $\mathcal{P}$ lying in the interior of the convex hull of $\mathcal{P}$. Without loss of generality we may assume that $q_i \in \triangle p_{2i-2}p_{2i-1}p_{2i} \cap \triangle p_{2i-1}p_{2i}p_{2i+1}$ for $i = 1, 2, 3$ (do not forget that no $\triangle p_{i-1}p_ip_{i+1}$ is empty, $1 \le i \le 6$). Then $p_1q_1q_3p_6$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by the line p_2p_5. Now we are done, since the set of the remaining five points necessarily contains an empty convex quadrangle, which is, of course, disjoint from $p_1q_1q_3p_6$.

Case 2. $m = 5$. Let q_1, q_2, q_3, q_4 denote the points of $\mathcal{P}$ lying in the interior of the convex hull of $\mathcal{P}$. A point of $\{q_1, q_2, q_3, q_4\}$ will be called special if it is contained in $\triangle p_{i-2}p_{i-1}p_i \cap \triangle p_{i-1}p_ip_{i+1}$ for some $1 \le i \le 5$. Obviously, at least one point of $\{q_1, q_2, q_3, q_4\}$ is special.

Case 2.1. Exactly one point of $\{q_1, q_2, q_3, q_4\}$, say q_4, is special. Without loss of generality we may assume that $q_4 \in \triangle p_1p_4p_5 \cap \triangle p_3p_4p_5$ and $q_i \in \triangle p_{i-2}p_ip_{i+2} \cap \triangle p_{i-1}p_ip_{i+1}$ for $i = 1, 2, 3$. Now $p_1q_1q_2p_2$ and $p_3q_3q_4p_4$ are disjoint empty convex quadrangles (they are separated by the line joining p_5 and $\overline{p_1p_3} \cap \overline{p_2p_4}$).

Case 2.2. Exactly two points of $\{q_1, q_2, q_3, q_4\}$, say q_1, q_2, are special. Then $\{q_1, q_2\} \subseteq \triangle p_{i-1}p_ip_{i+1}$ for at most one $1 \le i \le 5$.

Case 2.2.1. For some $1 \le i \le 5$, the set $\{q_1, q_2\} \subseteq \triangle p_{i-1}p_ip_{i+1}$. Without loss of generality we may assume that $q_j \in \triangle p_{j-2}p_{j-1}p_j \cap \triangle p_{j-1}p_jp_{j+1}$ for $j = 1, 2$ and $q_j \in \triangle p_{j-2}p_jp_{j+2} \cap \triangle p_{j-1}p_jp_{j+1}$ for $j = 3, 4$. Now $p_3p_4q_4q_3$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_2p_5.

Case 2.2.2. No $\triangle p_{i-1}p_ip_{i+1}$ contains both q_1 and q_2, $1 \le i \le 5$. Without loss of generality we may assume that $q_1 \in \triangle p_1p_2p_5 \cap \triangle p_1p_2p_3$, $q_2 \in \triangle p_2p_3p_4 \cap \triangle p_3p_4p_5$, and $q_3 \in \triangle p_1p_4p_5 \cap \triangle p_2p_3p_5$. Let $u = p_2q_3 \cap \overline{p_4p_5}$ and $v = p_3q_3 \cap \overline{p_5p_1}$ (see Figure 1).

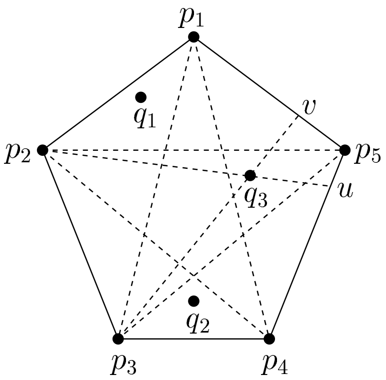

Figure 1.

If q_4 is contained in the quadrangle $p_2p_3p_4u$, then $p_1p_5q_3q_1$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_2q_3. Similarly, if q_4 is contained in the quadrangle $p_1p_2p_3v$, then $p_4p_5q_3q_2$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_3q_3. Finally, if q_4 is contained in the quadrangle up_5vq_3, then $p_2p_3q_2q_1$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_1p_4.

Case 2.3. Exactly three points of $\{q_1, q_2, q_3, q_4\}$, say q_1, q_2, q_3, are special.

Case 2.3.1. For some $1 \le i \le 5$, $\triangle p_{i-2}p_{i-1}p_i \cap \triangle p_{i-1}p_ip_{i+1}$ contains two points of $\{q_1, q_2, q_3\}$. Without loss of generality we may assume that $\{q_1, q_2\} \subseteq \triangle p_1p_4p_5 \cap \triangle p_1p_2p_5$, $q_3 \in \triangle p_1p_2p_3 \cap \triangle p_2p_3p_4$, and $q_4 \in \triangle p_2p_4p_1 \cap \triangle p_3p_4p_5$. Now $p_3p_4q_4q_3$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_2p_5.

Case 2.3.2. No $\triangle p_{i-2}p_{i-1}p_i \cap \triangle p_{i-1}p_ip_{i+1}$ contains two points of $\{q_1, q_2, q_3\}$, $1 \le i \le 5$. Then, among the triangles $\triangle p_{i-1}p_ip_{i+1}$, $1 \le i \le 5$, one or two contain two points of $\{q_1, q_2, q_3\}$.

Case 2.3.2.1. Among the triangles $\triangle p_{i-1}p_ip_{i+1}$, $1 \le i \le 5$, exactly one contains two points of $\{q_1, q_2, q_3\}$. Without loss of generality we may assume that $q_j \in \triangle p_{j-1}p_jp_{j+1} \cap \triangle p_jp_{j+1}p_{j+2}$ for $j = 1, 2$, $q_3 \in \triangle p_3p_4p_5 \cap \triangle p_4p_5p_1$, and q_4 is not separated from q_2 by the line p_2q_3. Now $p_1p_5q_3q_1$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_2q_3.

Case 2.3.2.2. Among the triangles $\triangle p_{i-1}p_ip_{i+1}$, $1 \le i \le 5$, exactly two contain two points of $\{q_1, q_2, q_3\}$. Without loss of generality we may assume that $q_j \in \triangle p_{j-1}p_jp_{j+1} \cap \triangle p_jp_{j+1}p_{j+2}$ for $j = 1, 2, 3$, and $q_4 \in \triangle p_2p_3p_5 \cap \triangle p_1p_4p_5$. Now $p_1p_5q_4q_1$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_2p_4.

Case 2.4. All four points of $\{q_1, q_2, q_3, q_4\}$ are special. Then there are three points of $\{q_1, q_2, q_3, q_4\}$, say q_1, q_2, q_3, so that no $\triangle p_{i-2}p_{i-1}p_i \cap \triangle p_{i-1}p_ip_{i+1}$ contains more than one point of $\{q_1, q_2, q_3\}$, $1 \leq i \leq 5$. Without loss of generality we may assume that $q_j \in \triangle p_{j-1}p_jp_{j+1} \cap \triangle p_jp_{j+1}p_{j+2}$ for $j = 1, 2$.

Case 2.4.1. The point q_3 is in $\triangle p_3p_4p_5 \cap \triangle p_4p_5p_1$. Without loss of generality we may assume that q_4 is not separated from q_2 by the line p_2q_3. Now $p_1p_5q_3q_1$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_2q_3.

Case 2.4.2. The point q_3 is contained in $\triangle p_2p_3p_4 \cap \triangle p_3p_4p_5$ or $\triangle p_4p_5p_1 \cap \triangle p_5p_1p_2$, say in $\triangle p_2p_3p_4 \cap \triangle p_3p_4p_5$. Then q_4 is contained in $\triangle p_3p_4p_5 \cap \triangle p_4p_5p_1$ or $\triangle p_4p_5p_1 \cap \triangle p_5p_1p_2$, say in $\triangle p_3p_4p_5 \cap \triangle p_4p_5p_1$ (do not forget that $\triangle p_1p_4p_5$ is not empty). Now $p_1p_5q_4q_1$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_2p_4.

Case 3. $m = 4$. Let q_1, q_2, q_3, q_4, q_5 denote the points of $\mathcal{P}$ lying in the interior of the convex hull of $\mathcal{P}$. Let $u = \overline{p_1p_3} \cap \overline{p_2p_4}$.

Case 3.1. No q_1, q_2, q_3, q_4, q_5 is contained in $\triangle p_iup_{i+1}$ for some $1 \leq i \leq 4$, say in $\triangle p_1up_2$. Without loss of generality we may assume that $\measuredangle q_1p_1p_2 < \measuredangle q_jp_1p_2$ for $2 \leq j \leq 5$, and $\measuredangle p_1q_1q_2 < \measuredangle p_1q_1q_k$ for $3 \leq k \leq 5$. Now $q_1 \in \triangle p_2up_3$ and the line q_1q_2 intersects $\overline{p_1p_4}$ since $\triangle p_1p_2p_3$ and $\triangle p_1p_2p_4$ are not empty. Then $p_1p_2q_1q_2$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by q_1q_2.

Case 3.2. All $\triangle p_iup_{i+1}$ contains at least on point of $\{q_1, q_2, q_3, q_4, q_5\}$, $1 \leq i \leq 4$. Without loss of generality we may assume that $q_i \in \triangle p_iup_{i+1}$ for $1 \leq i \leq 4$ and $q_5 \in \triangle p_4up_1$ (see Figure 2).

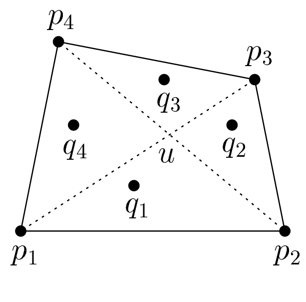

Figure 2.

Case 3.2.1. The line q_4q_5 does not intersect $\overline{p_1p_4}$. If q_4q_5 separates both q_1 and q_3 from p_1 and p_4, then $p_1p_4q_4q_5$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by q_4q_5. If q_4q_5 does not separate q_3 and q_1 from p_1 and p_4, then $p_1p_4q_1q_3$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by q_4q_5. Finally, if exactly one point of $\{q_1, q_3\}$, say q_1, is separated from p_1 and p_4 by q_4q_5, then $p_1p_4q_4q_3$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by q_4q_5.

Case 3.2.2. The line q_4q_5 intersects $\overline{p_1p_4}$. Without loss of generality we may assume that q_4q_5 is disjoint from $\triangle p_1up_2$. Now $p_1q_1q_4q_5$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_2p_4.

Case 4. $m = 3$. Let $q_1, q_2, q_3, q_4, q_5, q_6$ denote the points of $\mathcal{P}$ lying in the interior of the convex hull of $\mathcal{P}$. Without loss of generality we may assume that $\measuredangle p_2p_1q_1 < \measuredangle p_2p_1q_i$ for $2 \le i \le 6$ and $\measuredangle p_1q_1q_2 < \measuredangle p_1q_1q_i < \measuredangle p_1q_1q_6$ for $3 \le i \le 5$. Let $u = p_1q_1 \cap \overline{p_2p_3}$ and $v = p_2q_1 \cap \overline{p_1p_3}$ (see Figure 3).

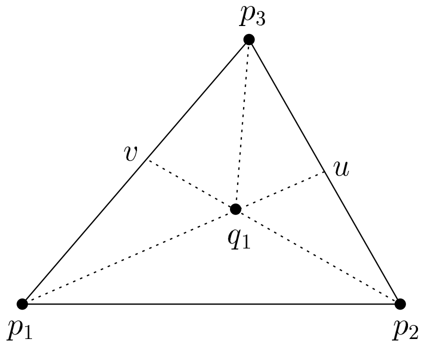

Figure 3.

It is obvious that $\triangle p_1up_2$ is empty. If $\triangle p_1q_1v$ is not empty, then $p_1p_2q_1q_2$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by q_1q_2. Thus, in what follows, we will assume that $\triangle p_1q_1v$ is empty. If $\triangle q_1p_3v$ is empty, then $p_1p_3q_2q_1$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by q_1q_2. Thus, in what follows, we will also assume that $\triangle q_1p_3v$ is not empty. Similarly, if $\triangle q_1p_3u$ is empty, then $p_2p_3q_6q_1$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by q_1q_6. Thus, in what follows, we will also assume that $\triangle q_1p_3u$ is not empty. For technical reason, in the remaining part of the proof we will disregard the special choice of q_2 and q_6.

Case 4.1. Exactly one point of $\{q_2, q_3, q_4, q_5, q_6\}$ is contained in $\triangle q_1p_3v$ or $\triangle q_1p_3u$. Without loss of generality we may assume that q_2 is contained in $\triangle q_1p_3u$ and q_3, q_4, q_5, q_6 are contained in $\triangle q_1p_3v$.

Case 4.1.1. Not all q_3, q_4, q_5, q_6 are separated from p_3 by p_2q_2. Without loss of generality we may assume that $\measuredangle p_3q_2q_3 < \measuredangle p_3q_2q_i$ for $4 \le i \le 6$.

Now $p_2p_3q_3q_2$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by q_2q_3.

Case 4.1.2. All q_3, q_4, q_5, q_6 are separated from p_3 by p_2q_2. Suppose that p_1p_3 and q_1q_2 are not parallel (the case where p_1p_3 and q_1q_2 are parallel can be settled similarly). Let $w = p_1p_3 \cap q_1q_2$. Without loss of generality we may assume that $\measuredangle q_1wq_3 < \measuredangle q_1wq_i$ for $4 \leq i \leq 6$ (see Figure 4).

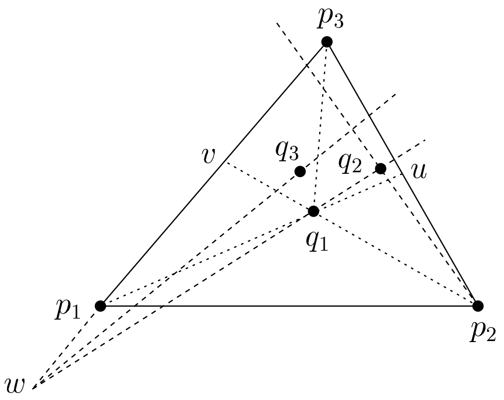

Figure 4.

Now $p_2q_1q_3q_2$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by wq_3.

Case 4.2. Exactly two points of $\{q_2, q_3, q_4, q_5, q_6\}$ are contained in $\triangle q_1p_3v$ or $\triangle q_1p_3u$. Without loss of generality we may assume that q_2, q_3 are contained in $\triangle q_1p_3u$ and q_4, q_5, q_6 are contained in $\triangle q_1p_3v$.

Case 4.2.1. The line q_2q_3 does not intersects $\overline{p_2p_3}$. Now $p_2p_3q_3q_2$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_3q_1.

Case 4.2.2. The line q_2q_3 intersects $\overline{p_2p_3}$ and $\overline{q_1p_3}$. Now $p_2q_3q_2q_1$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_3q_1.

Case 4.2.3. The line q_2q_3 intersects $\overline{p_2p_3}$ and $\overline{q_1p_2}$. Without loss of generality we may assume that q_3 separates q_2 from $q_2q_3 \cap \overline{p_2p_3}$ (see Figure 5). If $p_1q_4q_5q_6$ is a convex quadrangle then we are done. Indeed, now $p_1q_4q_5q_6$ is empty and it is separated from the remaining five points of $\mathcal{P}$ by p_3q_1. Thus, in what follows, we will assume, without loss of generality, that q_4 is contained in $\triangle p_1q_5q_6$. We will also assume that q_6 separates q_5 from $q_5q_6 \cap \overline{p_1p_3}$ if q_5q_6 intersects $\overline{p_1p_3}$ and that q_6 separates q_5 from $q_5q_6 \cap \overline{p_3q_1}$ if q_5q_6 does not intersect $\overline{p_1p_3}$.

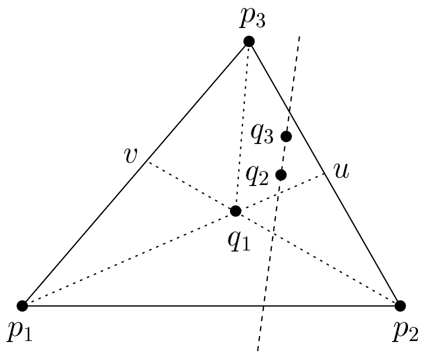

Figure 5.

Case 4.2.3.1. The line q_5q_6 separates p_3 from q_2 and q_3. If q_5q_6 does not intersect $\overline{p_1p_3}$, then $p_1p_3q_6q_4$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by q_5q_6. On the other hand, if q_5q_6 intersects $\overline{p_1p_3}$, then either $p_1p_3q_6q_4$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by q_4q_6 or $p_3q_6q_4q_5$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_1q_5.

Case 4.2.3.2. The line q_5q_6 separates q_2 and q_3. If q_5q_6 intersects $\overline{p_1p_3}$, then $p_1q_1q_2q_4$ and $p_3q_3q_5q_6$ are disjoint empty convex quadrangles. On the other hand, if q_5q_6 does not intersect $\overline{p_1p_3}$, then $p_3q_4q_6q_3$ and either $p_1q_1q_2q_5$ or $p_2q_1q_5q_2$ are disjoint empty convex quadrangles.

Case 4.2.3.3. The line q_5q_6 does not separate p_3 from q_2 and q_3. If q_5q_6 does not intersect $\overline{p_1p_3}$, then $p_2q_1q_5q_6$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by q_5q_6. If q_5q_6 intersects $\overline{p_1p_3}$ and $\overline{p_3q_1}$, then $p_3q_6q_2q_3$ and $p_1q_4q_5q_1$ are disjoint empty convex quadrangles. Thus, we will assume that q_5q_6 intersects $\overline{p_1p_3}$ and $\overline{p_1q_1}$. If $p_1p_3q_6q_4$ is a convex quadrangle, then we are done. Indeed, now $p_1p_3q_6q_4$ is empty and it is separated from the remaining five points of $\mathcal{P}$ by p_3q_6. Thus, we will assume that $p_3q_6q_4q_5$ is an empty convex quadrangle. If p_3 and q_3 are separated by p_1q_5, then $p_3q_6q_4q_5$ is separated from the remaining five points of $\mathcal{P}$ by p_1q_5. On the other hand, if p_3 and q_3 are not separated by p_1q_5, then $p_3q_6q_4q_3$ is an empty convex quadrangle, and either $p_1q_1q_2q_5$ or $p_2q_1q_5q_2$ is an empty convex quadrangle disjoint from $p_3q_6q_4q_3$.

Thus we have proved that any set $\mathcal{P}$ of nine points in general position in the plane contains two disjoint empty convex quadrangles.

The proof of Theorem 1 will be done by induction on n. We know that the assertion is true for $n \leq 9$. Let $n \geq 10$ and consider a set $\mathcal{P}$ of n points in general position. It is obvious that there exists a line which cuts $\mathcal{P}$ into two disjoint set $\mathcal{P}_1$ and $\mathcal{P}_2$ of 9 and $n-9$ points, respectively. Then, by the induction hypothesis, $\mathcal{P}_1$ contains two disjoint empty convex quadrangles and $\mathcal{P}_2$ contains $[2(n-9)/9] = [2n/9]-2$ disjoint empty convex quadrangles. Thus $\mathcal{P}$ contains $[2n/9]$ disjoint empty convex quadrangles.

3. Constructions

It is easy to find a set of eight points in general position in the plane which does not contain two disjoint empty convex quadrangles. Indeed, if p_1, p_2, p_3, p_4 are the vertices of a square in a counterclockwise order and q_i is an interior point of $p_1p_2p_3p_4$ sufficiently close to the midpoint of $\overline{p_ip_{i+1}}$, $1 \le i \le 4$, then $\{p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4\}$ is just such a point set.

Next we show that, for each $m \ge 3$, there exists a set of $n = 4m+1$ points in general position which does not contain m disjoint empty convex quadrangles.

Let $p_1, p_2, \dots, p_{2m}$ be the vertices of a regular $2m$-gon C in a counterclockwise order and let q_i is an interior point of C sufficiently close to the midpoint of $\overline{p_ip_{i+1}}$, $1 \le i \le 2m$. Furthermore, let r be a point sufficiently close to the centre of C so that $\mathcal{P} = \{p_1, \dots, p_{2m}, q_1, \dots, q_{2m}, r\}$ is in general position. Suppose, for contradiction, that $\mathcal{P}$ contains m disjoint empty convex quadrangles $Q_1, Q_2, \dots, Q_m$.

Let p_{i_1} and p_{i_2} two vertices of C so that they belong to Q_i for some $1 \le i \le m$ and the number l of vertices of C on the shorter arc of C bounded by p_{i_1} and p_{i_2} is as small as possible. Now, a very simple counting argument shows that $l \le 4$. Thus, without loss of generality we may assume that $i_1 = 1$ and $i_2 \in \{2,3,4\}$. If $i_2 = 2$, then q_1 cannot be a vertex of a quadrangle. If $i_2 = 3$, then p_2 cannot be a vertex of a quadrangle. Finally, if $i_2 = 4$, then p_2 or p_3 cannot be a vertex of a quadrangle. Next, let p_{j_1} and p_{j_2} two vertices of the longer arc of C bounded by p_{i_1} and p_{i_2} so that they belong to Q_j for some $1 \le j \le m$ and the number l' of vertices of C on the arc of C bounded by p_{j_1} and p_{j_2} is as small as possible. Again, a very simple counting argument shows that $l' \le 4$ and, similarly as before, we can find a point of $\mathcal{P}$ different from q_1, p_2, p_3 which cannot be a vertex of a quadrangle. Thus $\mathcal{P}$ necessarily contains two points which are not vertices of the quadrangles $Q_1, Q_2, \dots, Q_{2m}$, a contradiction.

Note that the above constructions show that the bound in Theorem 1 is tight for $n \le 21$.

References

[1] Bárány, I., Füredi, Z.: Empty simplices in Euclidean space. *Canad. Math. Bull.* **30** (1987), 436-445.

[2] Chung, F. R. K., Graham, R. L.: Forced convex n-gons in the plane. *Discrete Comput. Geom.* **19** (1998), 367-371.

[3] Erdős, P., Szekeres, Gy.: A combinatorial problem in geometry. *Compositio Math.* **2** (1935), 463-470.

[4] Harborth, H.: Konvexe Fünfecke in ebenen Punktmengen. *Elem. Math.* **33** (1978), 116-118.

[5] Horton, J. D.: Sets with no empty convex 7-gons. *Canad. Math. Bull.* **26** (1983), 482-484.

[6] Kleitman, D. J., Pachter, L.: Finding convex sets among points in the plane. *Discrete Comput. Geom.* **19** (1998), 405-410.

[7] Tóth, G., Valtr, P.: Note on the Erdős-Szekeres theorem. *Discrete Comput. Geom.* **19** (1998), 457-459.

[8] Valtr, P.: On the minimum number of empty polygons in planar point sets. *Studia Sci. Math. Hungar.* **30** (1995), 155-163.