On the number of cylinders touching a ball

Aladár Heppes
Computer and Automation Institute
Hungarian Academy of Sciences
Lágymányosi út 11
H-1111 Budapest
Hungary

László Szabó
Eötvös Loránd University
Department of Geometry
Rákóczi út 5
H-1088 Budapest
Hungary

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Abstract

The following problem is due to W. Kuperberg. What is the maximum number of nonoverlapping infinite unit cylinders in $\mathbb{E}^3$ that can be simultaneously tangent to a unit ball? In this paper we prove that this number is at most 8. It is conjectured that this number is 6.

1. Introduction

What is the maximum number τ_n of nonoverlapping unit balls in the n-dimensional Euclidean space that can be simultaneously tangent to a further unit ball?

Though the answer is trivial for the plane, $\tau_2 = 6$, it is well known that once the exact value of τ_3 was a point of controversy between Isaac Newton and David Gregory. Newton conjectured that the answer was 12 while Gregory thought 13 was also possible. It took almost two hundred years before R. Hoppe proved that Newton's guess, the '12', was correct [3]. After this it should not be surprising that today τ_n is known only for two further dimensions. Odlyzko and Sloane [6] proved that $\tau_8 \leq 240$ and $\tau_{24} \leq 196560$. Together with the fact that in the densest lattice packing of $\mathbb{E}^8$ with congruent balls each ball has 240 neighbours and the fact that the Leech lattice in $\mathbb{E}^{24}$ generates a ball packing with the 196560-neighbour property [5] this implies that $\tau_8 = 240$ and $\tau_{24} = 196560$. Moreover, the arrangements of balls producing these numbers are unique [1]. In this paper we consider a variant of this problem due to Kuperberg [4].

The set of points in the n-dimensional Euclidean space whose distance from a given line does not exceed a given positive number is called a cylinder. The given line is the cylinder's axis and the given number is its radius. The problem to be discussed here is to find the maximum number ϑ_n of nonoverlapping unit cylinders, i.e. cylinders of radius 1, in the n-dimensional Euclidean space, so that each cylinder is tangent to the same unit ball.

Clearly, ϑ_3 is at least 6 and it is very likely that this is the correct answer. Unfortunately, the problem turns out to be much harder than one would think at first sight. More difficulties can be expected, however, if one realizes that there are two totally different constructions using six cylinders. The first and trivial example is the one that consists of parallel cylinders touching a ball at the vertices of a regular hexagon inscribed in a great circle. The second example is a set of three pairs of parallel cylinders that are tangent to a ball at the three pairs of opposite vertices of an inscribed regular octahedron and are parallel to the three axes of this octahedron, respectively. Both examples are 'tight' in the sense that no single cylinder can be replaced with a different one.

The two authors of this paper studied this problem simultaneously and independently. Although they used different approaches and reached somewhat different results they thought it practical to publish the two methods in a common paper. While the first author considered the problem only in $\mathbb{E}^3$, he studied it in a more general form, replacing the unit cylinders with cylinders of radius r, and looking for the analogous number $\vartheta_3(r)$, the second author gave upper bounds for ϑ_3 and ϑ_4 as well. The first method works with area estimates for the 'central shadows' of the cylinders, while the second one operates with a generalization of the density estimation method due to Blichfeldt [2]. For 3-dimensional unit cylinders, both method give the same estimate, $\vartheta_3 \leq 8$.

2. The method of shadows

As mentioned above, instead of the original problem the following more general question will be studied in this section: What is the maximum number $\vartheta_3(r)$ of nonoverlapping cylinders of radius r in the 3-dimensional Euclidean space that can be simultaneously tangent to a further unit ball?

It is easy to see that for $r > 2\sqrt{3}+3$, $\vartheta_3(r) = 2$ and $\vartheta_3(2\sqrt{3}+3) = 3$. Moreover, if three nonoverlapping cylinders of radius $2\sqrt{3}+3$ are tangent to a unit ball then the cylinders must be parallel and tangent to each other. On the other hand, for sufficiently small r, the parallel arrangement of cylinders does not produce the maximum number $\vartheta_3(r)$. Indeed, a more detailed study shows that, as r decreases, the number of cylinders in the second 'octagonal' construction tends to one and the half times the number of cylinders in the first one. In the remainder of this section an upper bound will be given for the number of the cylinders as a function of their (common) radius. The upper bound is based on an estimate of the area of the spherical domain on the surface of the unit sphere from which a single cylinder can be (radially) 'seen'.

Lemma 1. *Let C_1 and C_2 be two disjoint open cylinders of radius r, each tangent to the unit ball around the origin O, and let P be a point in C_1 such that P is closer to O than $R = \sqrt{1+2r}$. Then the segment $\overline{OP}$ and the other cylinder C_2 are disjoint.*

Proof. Consider the (oriented) line segment starting at O and ending in P and its orthogonal projection on the plane ρ through O that is orthogonal to C_2. We assume that the segment $\overline{OP}$ and C_2 are not disjoint and we denote by P_1 and P_2 the points where $\overline{OP}$ enters and leaves C_2, finally, by P_1' and P_2' their projections on ρ, respectively. (In the limiting case P_1 and P_2 coincide.) Clearly, for the lengths of the segments it holds that $\overline{OP_1'} \cdot \overline{OP_2'} = R^2$ and $\overline{OP_1'} \leq \overline{OP_2'}$. Thus, whenever $\overline{OP}$ enters and leaves C_2 before entering C_1, the point P has to be at least as far from O than P_2 is, consequently, its distance from O is not less than R. $\square$

An immediate consequence of Lemma 1 is that the 'shadows' — orthogonal projections on the surface of the unit sphere — of those parts of the cylinders that are closer to O than R are disjoint. Thus, for the area $A(r)$ of such a shadow and for the number k of the cylinders the following inequality holds:

$$k \leq \frac{4\pi}{A(r)}.$$

Unfortunately, this estimate is not sharp enough to give exact answer to our question, therefore it did not seem to be worthwhile calculating $A(r)$ exactly. For the orientation of the reader we give a lower estimate — instead of the exact calculation of the area — of the function for some selected values of r. In the table below each quarter of the (convex) shadow has been estimated from below by the area of an inscribed quadrangle.

r	$A(r) >$
1.275	1.7957
1.075	1.5741
0.960	1.3973

From this follows

Theorem 1.

$$\vartheta_3(0.960) \leq 8,$$
$$\vartheta_3(1.075) \leq 7,$$
$$\vartheta_3(1.275) \leq 6.$$

3. The generalized Blichfeldt method

In this section we generalize the following result of Blichfeldt [2].

Lemma 2. *Let $\mathcal{F}$ be a family of unit balls in the n-dimensional Euclidean space. Suppose that the interiors of the members of $\mathcal{F}$ are disjoint. Leaving the centres fixed, we replace the given geometrical balls with physical balls of superposable matter, all of radius $\sqrt{2}$. Furthermore, we assume that the density of the matter in each ball equals $2-r^2$ at a distance r from its centre. Then the density of the matter at no point in $\mathbb{E}^n$ is greater than 2.*

We shall extend this result to the case when the packing consists of unit balls and unit cylinders.

Lemma 3. *Let $\mathcal{F}$ be a family of unit balls $B_1, B_2, \ldots$ and unit cylinders $C_1, C_2, \ldots$ in the n-dimensional Euclidean space. Suppose that the interiors of the members of $\mathcal{F}$ are disjoint. Leaving the centres fixed, we replace the given geometrical balls $B_1, B_2, \ldots$ with physical balls $B_1', B_2', \ldots$ of superposable matter, all of radius $\sqrt{2}$. Furthermore, we assume that the density of the matter in each ball equals $2-r^2$ at a distance r from its centre. Similarly, leaving the axes fixed, we replace the given geometrical cylinders $C_1, C_2, \ldots$ with physical cylinders $C_1', C_2', \ldots$ of superposable matter, all of radius $\sqrt{2}$.*

Furthermore, we assume that the density of the matter in each cylinder equals $2 - r^2$ at a distance r from its axis. Then the density of the matter at no point in $\mathbb{E}^n$ is greater than 2.

Proof. Let P be an arbitrary point in $\mathbb{E}^n$. Let $P_1, P_2, \dots$ denote the orthogonal projections of P onto the axes of $C_1, C_2, \dots$ and let $D_1, D_2, \dots$ be the unit balls centered at $P_1, P_2, \dots$, respectively. Since the new balls $D_1, D_2, \dots$ are inside the cylinders $C_1, C_2, \dots$, consequently the balls $B_1, B_2, \dots, D_1, D_2, \dots$ are disjoint. Now we replace the geometrical balls $B_1, B_2, \dots, D_1, D_2, \dots$ with physical balls $B_1', B_2', \dots, D_1', D_2', \dots$ as described above. Obviously, the density of the matter at the point P does not change. Therefore, by Lemma 2, the density of the matter at the point P does not exceed 2. $\square$

Now suppose that k nonoverlapping unit cylinders in the n-dimensional Euclidean space are tangent to a unit ball. Replace the geometrical cylinders with the physical cylinders $C_1, C_2, \dots, C_k$ and the geometrical ball with the physical ball B as described in Lemma 3. Let B^* denote the ball of radius $\sqrt{7}$ concentric with B. By Lemma 3, the density of the matter in B^* is at most 2. The volume of B^* is

$$V = \frac{7^{n/2}\pi^{n/2}}{\Gamma\left(1 + \frac{n}{2}\right)}.$$

Furthermore, it is easy to check that the mass of B is

$$M_b = s_n \int_0^{\sqrt{2}} (2 - x^2)x^{n-1}dx = s_n \frac{2^{2+n/2}}{n(n+2)}$$

and the mass of $C_i \cap B^*$, $(i = 1, 2, \dots, k)$, is

$$M_c = \int_{-\sqrt{1+4\sqrt{2}}}^{\sqrt{1+4\sqrt{2}}} \left[\int_{2-\sqrt{7-z^2}}^{\frac{z^2-1}{4}} \int_0^{\sqrt{-x^2+4x+3-z^2}} s_{n-2}y^{n-3}(2 - x^2 - y^2)\, dy\, dx + \int_{\frac{z^2-1}{4}}^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} s_{n-2}y^{n-3}(2 - x^2 - y^2)\, dy\, dx \right] dz$$

where s_n is the surface area of the n-dimensional unit ball,

$$s_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}.$$

Hence we obtain an upper bound for k,

$$k \leq \frac{2V - M_b}{M_c}.$$

For $n = 3, 4$ the numerical values are

n	V	M_b	M_c	$k \leq$
3	77.5774	9.4781	17.3857	8.3792
4	241.8053	13.1594	25.0639	18.7701

This implies

Theorem 2.

$$\vartheta_3 \leq 8,$$
$$\vartheta_4 \leq 18.$$

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