Clouds of planar convex bodies

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Abstract

Let $\{K, K_1, \ldots, K_n\}$ be a family of convex bodies in the plane. We say that the bodies $K_1, \ldots, K_n$ form a deep cloud around K if the bodies $K, K_1, \ldots, K_n$ are mutually non-overlapping and every ray emanating from K intersects the interior of at least one of $K_1, \ldots, K_n$. In this paper we are concerned with the following question: What is the minimum number of congruent copies of a given convex body K forming a deep cloud around K?

1. Introduction

The following problem was posed by Károly Böröczky and Valeriu Soltan (see [3]). Let $\{K, K_1, \ldots, K_n\}$ be a family of convex bodies (i.e. compact convex sets with non-empty interior) in the d-dimensional Euclidean space. We say that the bodies $K_1, \ldots, K_n$ form a deep cloud around K if the bodies $K, K_1, \ldots, K_n$ are mutually non-overlapping and every ray emanating from K intersects the interior of at least one of $K_1, \ldots, K_n$. What is the minimum number $C_T(K)$ of translates of a given convex body K forming a deep cloud around K?

Károly Böröczky and Valeriu Soltan proved in [3] that $C_T(K)$ is finite for every d-dimensional convex body K. Shortly afterwards Chuanming Zong

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verified in [6] the upper bound d^{d^2} for every d-dimensional convex body, which was improved to c^{d^2} by István Talata in [5] and by Károly Böröczky Jr. and Gábor Tardos in [2]. More precisely, Károly Böröczky Jr. and Gábor Tardos proved in [2] that $C_T(K) \leq 4^{d^2+o(d^2)}$ for every d-dimensional convex body K and $C_T(K) \leq 2^{d^2+o(d^2)}$ for every d-dimensional centrally symmetric convex body K. With respect to the d-dimensional unit ball B^d, István Talata showed in [5] that $2^{0.599d^2-o(d^2)} \leq C_T(B^d) \leq 2^{1.401d^2+o(d^2)}$. We close this paragraph with a challenging open problem due to Valeriu Soltan (see [4]) concerning the planar version of the problem. Is it true that $8 \leq C_T(K) \leq 9$ for every planar convex body K?

In this paper we are concerned with the following variant of the above problem. What is the minimum number $C(K)$ of congruent copies of a given planar convex body K forming a deep cloud around K?

It is easy to see that $C(K) \geq 4$ for any planar convex body K. Indeed, the four rays emanating from the endpoints of a diameter of K and being perpendicular to this diameter intersect the interior of at least four different members of any deep cloud of congruent copies of K around K. It is also easy to see that this lower bound is attained in many cases, Figure 1 shows an example.

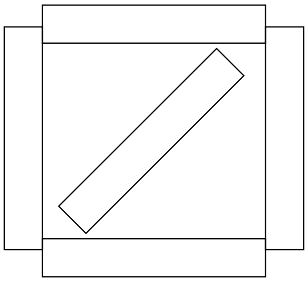


Figure 1.

In this paper we prove the following two theorems.

Theorem 1. *Let K be a planar convex body. Then $C(K) \leq 9$ with equality if and only if K is a circle.*

Theorem 2. *Let K be a planar convex body different from a circle. Suppose that there exist two concentric circles B and B' with radius $r = 0.99$ and $r' = 1$, respectively, such that $B \subseteq K \subseteq B'$. Then $C(K) = 8$.*

For additional problems and results concerning clouds of different kinds we refer the reader to the excellent survey paper by Horst Martini and Valeriu Soltan [4].

2. Proof of Theorem 1

First we prove that $C(K) \le 8$ for any planar convex body K different from a circle. For a diameter $\overline{UV}$ of K, let $R(K, \overline{UV}) = PQST$ denote the rectangle circumscribed around K such that PQ is parallel to UV and QS is perpendicular to UV. Note that $PT \cap K$ and $QS \cap K$ are single points.

If $|\overline{PQ}| > |\overline{QS}|$ in $R(K, \overline{UV})$ for some diameter $\overline{UV}$ of K, then consider the arrangement of eight copies $K_1, K_2, \ldots, K_8$ of K around K as shown in Figure 2a. It is important that the points of $K_1, K_2, \ldots, K_8$ lying on the boundary of their circumscribed rectangles with respect to the midpoints of the sides of these rectangles are as shown in Figure 2a. Now translate first K_3, K_7 by suitable small horizontal vectors and then K_2, K_4, K_6, K_8 by suitable small vertical vectors as shown in Figure 2b. It is easy to see that in this way we obtain an eight-element deep cloud around K.

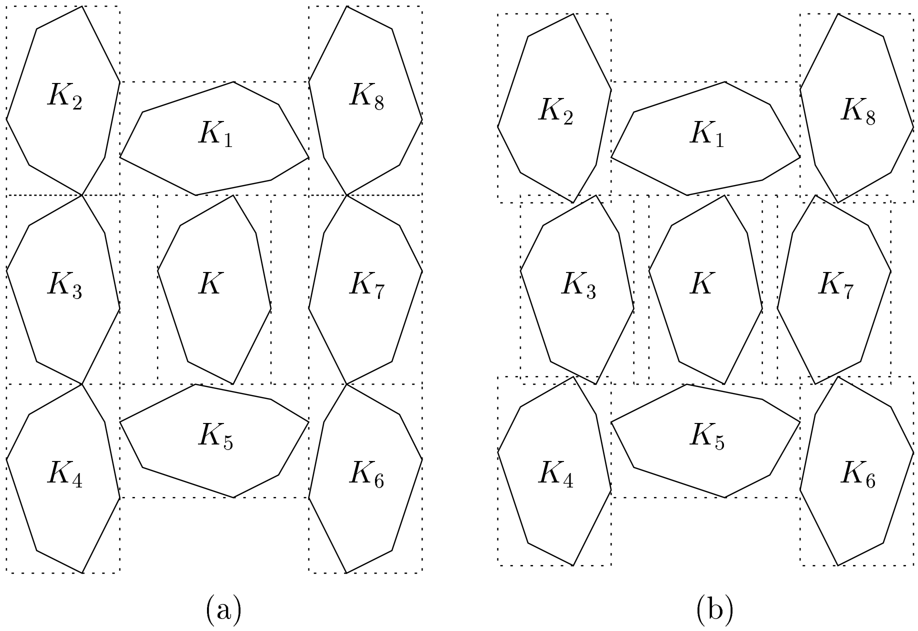


Figure 2.

In what follows, we assume that $|\overline{PQ}| = |\overline{QS}|$ in $R(K, \overline{UV})$ for every diameter $\overline{UV}$ of K. If there exists a rectangle containing K whose sides are smaller than $|\overline{UV}|$, then consider the arrangement of eight copies $K_1, K_2, \ldots, K_8$ of K around K as shown in Figure 3a. Observe that there exists a positive number ε such that the distances of the vertices of the rectangles circumscribed around $K_1, K_2, \ldots, K_8$ from $K_1, K_2, \ldots, K_8$, respectively, are at least ε. Now translate K_1, K_5 by suitable small vertical vectors and K_3, K_7 by suitable small horizontal vectors as shown in Figure 3b. It is easy to see that in this way we obtain an eight-element deep cloud around K.

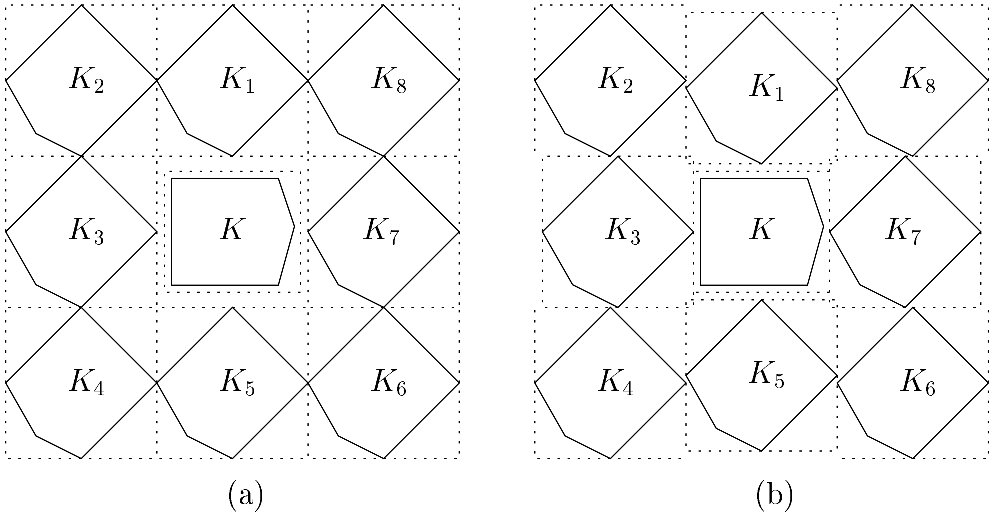
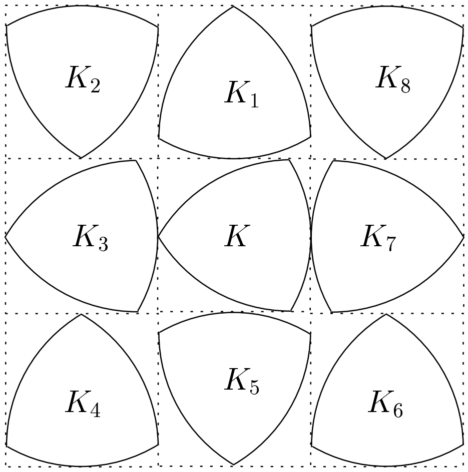


Figure 3.

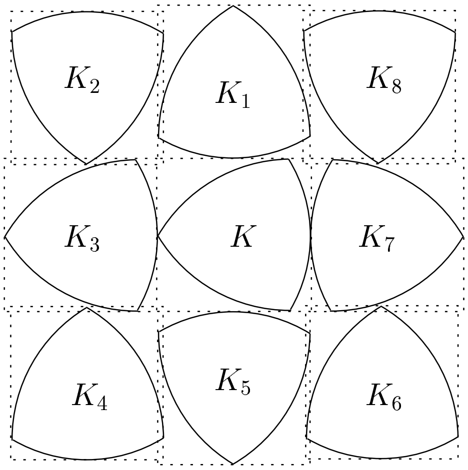
In what follows, we assume that every rectangle circumscribed around K is a square of side length $|\overline{UV}|$ where $\overline{UV}$ is a diameter of K. If U and V are not the midpoints of the sides $\overline{QS}$ and $\overline{PT}$ of $R(K, \overline{UV})$ for some diameter $\overline{UV}$ of K, then consider the arrangement of eight copies $K_1, K_2, \dots, K_8$ of K around K as shown in Figure 4a. It is important that the points of $K_1, K_2, \dots, K_8$ lying on the boundary of their circumscribed squares with respect to the midpoints of the sides of these squares are as shown in Figure 4a. Now translate K_2, K_4, K_6, K_8 first by suitable small vertical vectors and then by suitable small horizontal vectors as shown in Figure 4b. It is easy to see that in this way we obtain an eight-element deep cloud around K.

Finally, also assume that U and V are the midpoints of the sides $\overline{QS}$ and $\overline{PT}$ of $R(K, \overline{UV})$ for every diameter $\overline{UV}$ of K. We show that then K is a circle.

At this point let us recall a remarkable result of Károly Böröczky Jr. [1]. The width of a planar convex body S in the direction of a line l is the distance between the two parallel supporting lines of K that are perpendicular to l and that contain S between them. If the width of S is the same for all directions then S is said to be of constant width. Károly Böröczky Jr. proved in [1] that if for any two points X, Y of a planar convex body S of constant width there exists a non-degenerate rectangle $R \subseteq S$ such that $X, Y \in R$ then S is a circle.



(a)



(b)

Figure 4.

Since K is a body of constant width now, therefore, by the above mentioned result of Károly Böröczky Jr., it is enough to prove that for any two points X, Y of K there exists a non-degenerate rectangle $R \subseteq K$ such that $X, Y \in R$.

Let X and Y be two boundary points of K (obviously, it is enough to prove the assertion for such points). If there are two parallel supporting lines a and b of K going through X and Y, respectively, then the circumscribed rectangle around K whose sides are parallel and perpendicular to a is a square and X and Y are midpoints of its corresponding sides. Now, the points X, Y and the remaining two midpoints of the sides of this square are the vertices of a square which is, of course, contained in K.

Next assume that any two supporting lines of K going through X and Y, respectively, intersect each other. Let a and b be two arbitrary supporting lines of K going through X and Y, respectively, and let M be their common point.

We claim that $\measuredangle MXY, \measuredangle MYX < \pi/2$. Suppose, for contradiction, that $\measuredangle MYX \geq \pi/2$. Since K is a body of constant width, therefore there exist a point Z on the boundary of K such that $\overline{YZ}$ is a diameter of K. Now $\measuredangle MYZ = \pi/2$, hence Z is contained in the triangle MYX which is possible, however, only if $Z = X$. This implies that the lines perpendicular to XY and going through X and Y, respectively, are parallel supporting lines of K, a contradiction. The inequality $\measuredangle MXY < \pi/2$ can be shown similarly.

Thus, if any two supporting lines of K going through X and Y, respectively, intersect each other, then the lines perpendicular to XY and going through X and Y, respectively, contains interior points of K which are not separated by XY. It is obvious that now we can find a non-degenerate rectangle in K which contains $\overline{XY}$.

This finishes the proof of the inequality $C(K) \leq 8$ for any planar convex body K different from a circle.

To complete the proof of Theorem 1 we have to prove that $C(B) = 9$ for a circle B. Figure 5 shows that $C(B) \leq 9$. In Figure 5 the circles are of unit radius, two circle centres are joined by a segment if their distance is two, and $\measuredangle OO_1O_2 = \measuredangle OO_1O_9 = \measuredangle OO_3O_4 = \measuredangle OO_8O_7 = \frac{\pi}{2} + \varepsilon$ where ε is a sufficiently small positive number.

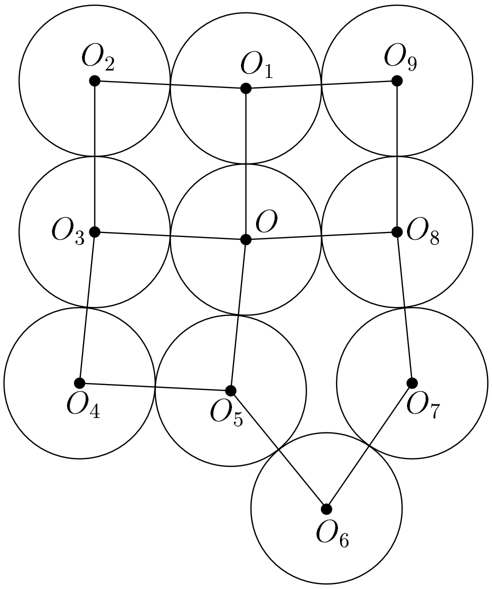


Figure 5.

It remains to prove that eight unit circles cannot form a deep cloud around a unit circle. Suppose, for contradiction, that the unit circles $B_1, \ldots, B_8$ form a deep cloud around the unit circle B. Let $O, O_1, \ldots, O_8$ denote the centres of $B, B_1, \ldots, B_8$, respectively. We may assume that the rays emanating from O and going through $O_1, \ldots, O_8$, respectively, are in a counterclockwise order. As usual, we will use the convention that $O_i = O_j$ if $i \equiv j \pmod 8$.

First of all, observe that $\measuredangle O_iOO_{i+1} < \pi/3$ for each $1 \leq i \leq 8$. Indeed, if $\measuredangle O_iOO_{i+1} \geq \pi/3$ for some $1 \leq i \leq 8$, then the angle bisector (half-line) of $\angle O_iOO_{i+1}$ intersects the interior of none of $B_1, \ldots, B_8$.

Proposition 3. *If* $|\overline{OO_i}| \leq 2\sqrt{2}$, *then* $|\overline{OO_{i+1}}| > 2\sqrt{2}$, $1 \leq i \leq 8$.

Proof. Suppose, for contradiction, that $|\overline{OO_i}|, |\overline{OO_{i+1}}| \leq 2\sqrt{2}$ for some $1 \leq i \leq 8$. Consider the perpendicular bisector l of the segment $\overline{O_iO_{i+1}}$. Without loss of generality we may assume that l does not separate O and O_i. Since $|\overline{OO_i}|, |\overline{O_iO_{i+1}}| \geq 2$ and $|\overline{OO_{i+1}}| \leq 2\sqrt{2}$, therefore $\measuredangle OO_iO_{i+1} \leq \pi/2$. Set $|\overline{O_iO_{i+1}}| = 2 + 2x$, where $x \geq 0$. If l does not intersect B, then

$$|\overline{OO_{i+1}}| > \sqrt{(4-x^2)+(2+x)^2} = \sqrt{8+2x} \geq 2\sqrt{2},$$

which is impossible. Thus l intersects B.

Now, the ray emanating from the orthogonal projection P of O onto l and going through the midpoint Q of $\overline{O_iO_{i+1}}$ must intersect the interior of a circle B_j for some $1 \leq j \leq 8$. It is clear that $j \neq i, i+1$. It is also clear that B_j intersects $\overline{PQ}$. If l does not separate O and O_j, then $\measuredangle OO_jO_i > \pi/2$, hence $|\overline{OO_i}| > 2\sqrt{2}$, a contradiction. On the other hand, if l separates O and O_j, then $\measuredangle OO_jO_{i+1} > \pi/2$, hence $|\overline{OO_{i+1}}| > 2\sqrt{2}$, a contradiction again. □

Consider the regions $\angle O_iOO_{i+1}$, $1 \leq i \leq 8$. Assume that $\measuredangle O_iOO_{i+1} \geq \pi/4$ for some $1 \leq i \leq 8$, say for $i = 8$. By Proposition 3, at least one of $|\overline{OO_1}| > 2\sqrt{2}$ and $|\overline{OO_8}| > 2\sqrt{2}$ holds. Assume that $|\overline{OO_8}| > 2\sqrt{2}$. Then the distance of O_8 from the line OO_1 is at least 2 (do not forget that $\measuredangle O_8OO_1 < \pi/3$). Let l be the common supporting line of B and B_1 which separates O_8 from O and O_1. Now the ray emanating from $B \cap l$ and going through $B_1 \cap l$ intersects the interior of a circle B_j for some $1 < j < 8$. Obviously, $\measuredangle O_jOO_1 < \pi/2$ and $|\overline{OO_j}| \leq 2\sqrt{2}$. Let $2 \leq k \leq 7$ be the maximal index such that $|\overline{OO_k}| \leq 2\sqrt{2}$. Then $\measuredangle O_kOO_1 < \pi/2$.

Next consider the segment $\overline{OO_1}$. If $|\overline{OO_1}| > 2\sqrt{2}$, then let $2 \leq m \leq 7$ be the minimal index such that $|\overline{OO_m}| \leq 2\sqrt{2}$. Now, similar argument as before shows that $\measuredangle O_8OO_m < \pi/2$. This implies that if $\measuredangle O_8OO_1 \geq \pi/4$, then there exist indices $1 \leq m \leq k \leq 8$ such that $\measuredangle O_kOO_m < (m+8-k)\frac{\pi}{4}$, $|\overline{OO_k}|, |\overline{OO_m}| \leq 2\sqrt{2}$, and $|\overline{OO_j}| > 2\sqrt{2}$ if $k < j \leq 8$ or $1 \leq j < m$.

Now, repeating the above argument for each region $\angle O_iOO_{i+1}$ such that $\measuredangle O_iOO_{i+1} \geq \pi/4$ we obtain that the average of the angles $\measuredangle O_iOO_{i+1}$, $1 \leq i \leq 8$, is strictly less than $\pi/4$, a contradiction. This completes the proof of the inequality $C(B) \geq 9$.

3. Proof of Theorem 2

It follows from Theorem 1 that $C(K) \leq 8$. The idea of the proof of the inequality $C(K) \geq 8$ is similar to the idea we used in the proof of the inequality $C(B) \geq 9$ in the previous section. Assume, for contradiction, that seven congruent copies $K_1, \ldots, K_7$ of K form a deep cloud around K. Let B and B' be two concentric circles with radius $r = 0.99$ and $r' = 1$, respectively, such that $B \subseteq K \subseteq B'$. Also, for $i = 1, \dots, 7$, let B_i and B_i' be two concentric circles with radius 0.99 and 1, respectively, such that $B_i \subseteq K_i \subseteq B_i'$. Let $O, O_1, \dots, O_7$ denote the centres of $B, B_1, \dots, B_7$, respectively. We may assume that the rays emanating from O and going through $O_1, \dots, O_7$, respectively, are in a counterclockwise order. Again, we will use the convention that $O_i = O_j$ if $i \equiv j \pmod 7$.

First of all, observe that $\measuredangle O_iOO_{i+1} < 2\arcsin\frac{r'}{2r}$ for each $1 \le i \le 7$. Indeed, if $\measuredangle O_iOO_{i+1} \ge 2\arcsin\frac{r'}{2r}$ for some $1 \le i \le 7$, then the angle bisector (half-line) of $\angle O_iOO_{i+1}$ intersects the interior of none of $K_1, \dots, K_7$.

Proposition 4. *If* $|\overline{OO_i}| \le 2.57$, *then* $|\overline{OO_{i+1}}| > 2.57$, $1 \le i \le 7$.

Proof. Suppose, for contradiction, that $|\overline{OO_i}|, |\overline{OO_{i+1}}| \le 2.57$ for some $1 \le i \le 7$. We note that then $\measuredangle OO_iO_{i+1}, \measuredangle OO_{i+1}O_i \le \pi/2$.

Case 1. $|\overline{O_iO_{i+1}}| \ge 2$. Consider the perpendicular bisector l of the segment $\overline{O_iO_{i+1}}$. Now l separates K_i and K_{i+1}. Without loss of generality we may assume that O and O_i are not separated by l. Set $|\overline{O_iO_{i+1}}| = 2 + 2x$, where $x \ge 0$ (see Figure 6).

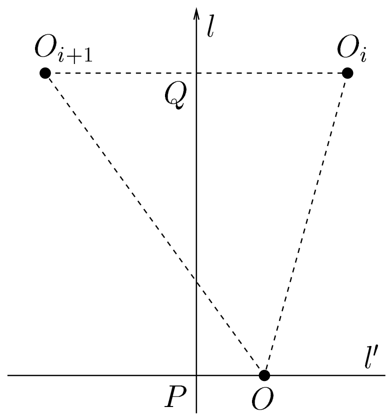


Figure 6.

If l does not intersect B, then

$$|\overline{OO_{i+1}}| > [(r' + x + r)^2 + (4r^2 - (r' + x - r)^2)]^{1/2} \ge [4rr' + 4r^2]^{1/2} > 2.8,$$

which is impossible. Thus l intersects B.

Now, the ray emanating from the orthogonal projection P of O onto l and going through the midpoint Q of $\overline{O_iO_{i+1}}$ must intersect the interior of a body K_j for some $1 \le j \le 7$. It is clear that $j \ne i, i+1$. It is also easy to see that K_j intersects $\overline{PQ}$.

If O and O_j are not separated by l then $\measuredangle OO_jO_i > \pi/2$, hence $|\overline{OO_i}| > 1.98\sqrt{2} > 2.57$, a contradiction. On the other hand, if O and O_j are separated by l, then $\measuredangle OO_jO_{i+1} > \arcsin 0.99$. Indeed, if the line l' perpendicular to l and going through O does not separate O_j and O_{i+1}, then it is obvious, and if l' separates O_j and O_{i+1}, then $|\overline{O_jP}| < r'$ and the distance of O_j from l is greater than r which imply that the angle between the lines OO_j and l is greater than $\arcsin 0.99$. Thus $|\overline{OO_{i+1}}| > 2.57$, a contradiction again.

Case 2. $1.98 \leq |\overline{O_iO_{i+1}}| < 2$. Let l be a line which separates K_i and K_{i+1}. Without loss of generality we may assume that O and O_i are not separated by l. Let $Q = l \cap O_iO_{i+1}$ and let T_i and T_{i+1} denote the orthogonal projections of O_i and O_{i+1}, respectively, onto l. Now $|\overline{O_iT_i}|, |\overline{O_{i+1}T_{i+1}}| \geq r$ and at least one of $|\overline{O_iQ}| \leq r'$ and $|\overline{O_{i+1}Q}| \leq r'$ holds. Thus $\measuredangle O_iQT_i = \measuredangle O_{i+1}QT_{i+1} \geq \arcsin \frac{r}{r'}$. Moreover, $|\overline{O_iT_i}|, |\overline{O_{i+1}T_{i+1}}| \leq 2r' - r = 1.01$ (see Figure 7).

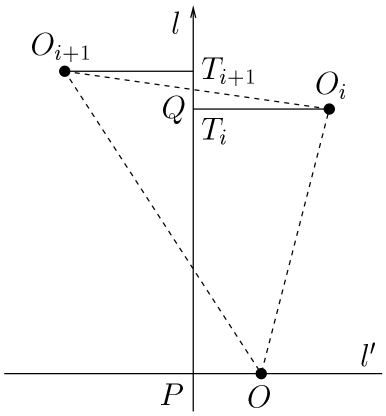


Figure 7.

If the distance of O from l is at least 0.98, then

$$\measuredangle OO_iO_{i+1} > \frac{\pi}{2} - \arccos \frac{r}{r'} - \arcsin \frac{2r' - r - 0.98}{2r} > 1.4141,$$

and thus

$$|\overline{O_iO_{i+1}}| \geq 4r \sin \frac{\measuredangle OO_iO_{i+1}}{2} > 2.5723,$$

which is impossible. Thus the distance of O from l is smaller than 0.98, hence l intersects B. Note that the angle between the lines OO_i and l is greater than

$$\arcsin \frac{2r' - r - 0.98}{2r} > 0.01515.$$

Now, the ray emanating from the orthogonal projection P of O onto l and going through Q must intersect the interior of a body K_j for some $1 \leq j \leq 7$.

It is clear that $j \neq i, i+1$. It is also easy to see that K_j intersects $\overline{PQ}$ (do not forget that the angle between the lines OO_i and l is at least 0.01515).

If O and O_j are not separated by l then $\measuredangle OO_jO_i > \pi/2$ hence $|\overline{OO_i}| > 1.98\sqrt{2} > 2.57$, a contradiction. On the other hand, if O and O_j are separated by l, then

$$\measuredangle OO_jO_{i+1} > \frac{\pi}{2} - \arccos\frac{r}{r'} - \arcsin\frac{r'-r}{2r} > 1.4242.$$

Indeed, if the line l' perpendicular to l and going through O does not separate O_j and O_{i+1}, then it is obvious, and if l' separates O_j and O_{i+1} then it is enough to observe that the angle between the lines OO_j and l is greater than $\arcsin 0.99$. Thus $|\overline{OO_{i+1}}| > 2.57$, a contradiction again. □

Consider the regions $\angle O_iOO_{i+1}$, $1 \leq i \leq 7$. Assume that $\measuredangle O_iOO_{i+1} \geq 2\pi/7$ for some $1 \leq i \leq 7$, say for $i = 7$. By Proposition 4, at least one of $|\overline{OO_1}| > 2.57$ and $|\overline{OO_7}| > 2.57$ holds. Assume that $|\overline{OO_7}| > 2.57$. Then the distance of O_7 from the line OO_1 is at least 2 (do not forget that $\measuredangle O_7OO_1 < 2\arcsin\frac{r'}{2r}$).

Let l^* be the line such that l^* goes through O, the angle between the lines l^* and OO_1 is $\arcsin\frac{r'-r}{2r}$, and l^* separates O_1 and O_7. Then the distance of O_7 from l^* is at least

$$2.57\sin\left(\frac{2\pi}{7} - \arcsin\frac{r'-r}{2r}\right) > 2.001.$$

Let l be the supporting line of K parallel to l^* which separates O and O_7. Then the distance between l and l^* is at least r and is at most r'. Thus the distance of O_7 from l is greater than r' (see Figure 8).

Let h denote the ray emanating from a point P of $K \cap l$ and going through $O_1O_7 \cap l$. Now h intersects the interior of a body K_j for some $1 \leq j \leq 6$. If l^* separates O_j and O_7, then the angle between the lines OO_j and l^* is at least $\arcsin\frac{r'-r}{2r}$ and thus the distance of O_j from l is at least

$$r + 2r\sin\left(\arcsin\frac{r'-r}{2r}\right) = r',$$

which is impossible. Therefore l^* does not separate O_j and O_7. Then OO_7 separates O_1 and O_j. Now $|\overline{OO_j}| \leq 2.57$.

We claim that

$$\measuredangle O_1OO_j \leq \arcsin\frac{r'-r}{2r} + \frac{\pi}{2} + \arccos\frac{r}{r'} < \frac{4\pi}{7}.$$

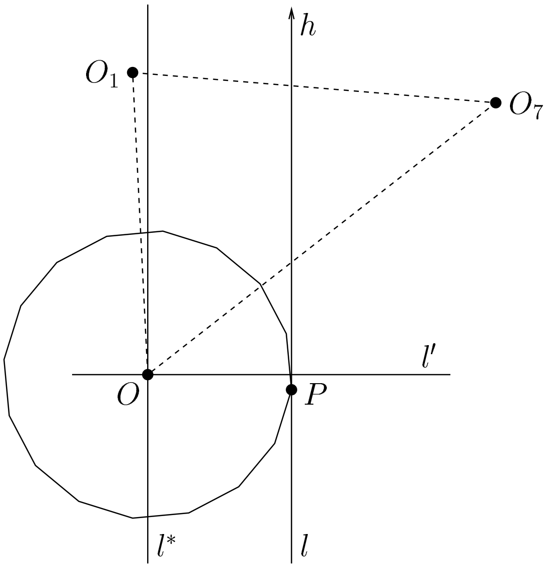


Figure 8.

Indeed, since $|\overline{OP}| \leq r'$ and the distance of O from l is at least r, therefore the angle between the lines OP and l is at least $\arcsin \frac{r}{r'}$. Now the inequality is obvious if the line OP or the line l' perpendicular to l and going through O does not separate O_j and O_7. On the other hand, if both OP and l' separate O_j and O_7, then $|\overline{O_jP}| \leq r'$ and the distance of O_j from l is at least r. Thus the angle between the lines OO_j and l is at least $\arcsin \frac{r}{r'}$ from which the inequality follows again.

Let $2 \leq k \leq 6$ be the maximal index such that $|\overline{OO_k}| \leq 2.57$. Then $\measuredangle O_kOO_1 < 4\pi/7$.

Next consider the segment $\overline{OO_1}$. If $|\overline{OO_1}| > 2.57$, then let $2 \leq m \leq 6$ be the minimal index such that $|\overline{OO_m}| \leq 2.57$. Now, similar argument as before shows that $\measuredangle O_7OO_m < 4\pi/7$. This implies that if $\measuredangle O_7OO_1 \geq 2\pi/7$, then there exist indices $1 \leq m \leq k \leq 8$ such that $\measuredangle O_kOO_m < (m+7-k)\frac{2\pi}{7}$, $|\overline{OO_k}|, |\overline{OO_m}| \leq 2.57$, and $|\overline{OO_j}| > 2\sqrt{2}$ if $k < j \leq 7$ or $1 \leq j < m$.

Now, repeating the above argument for each region $\angle O_iOO_{i+1}$ such that $\measuredangle O_iOO_{i+1} \geq 2\pi/7$ we obtain that the average of the angles $\measuredangle O_iOO_{i+1}$, $1 \leq i \leq 7$, is strictly less than $2\pi/7$, a contradiction. This completes the proof of the inequality $C(K) \geq 8$.

Note that the above proof immediately yields the following

Corollary 5. *Let $\{K, K_1, \ldots, K_n\}$ be a family of planar convex bodies such that $K_1, \ldots, K_n$ form a deep cloud around K. Suppose that there exist two concentric circles B and B' with radius* 0.99 *and* 1, *respectively, such that*

$B \subseteq K \subseteq B'$. Also, for $i = 1, 2, \ldots, n$, suppose that there exist two concentric circles B_i and B_i' with radius 0.99 and 1, respectively, such that $B_i \subseteq K_i \subseteq B_i'$. Then $n \geq 8$.

Acknowledgement. We would like to thank Ludwig Danzer for his long, stimulating letter consisting of many interesting questions, conjectures and specific constructions related to the problem discussed in this paper.

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