On the density of unit balls touching a unit cylinder

László Szabó
Eötvös Loránd University
Department of Geometry
Rákóczi út 5
H-1088 Budapest
Hungary

August 2, 2000

Abstract

In this paper we give reasonably good upper bounds for the density of unit balls touching an infinite unit cylinder without overlapping each other in the 4-, 6-, and 8-dimensional Euclidean spaces.

1. Introduction

The following problem was raised by Kuperberg [10]. What is the maximum number of infinite unit cylinders in E^3 that can touch a unit ball without overlapping each other? It is conjectured that the number is 6, but observe that the configuration of six unit cylinders touching a unit ball is not unique. It was shown in [7] that the number in question is not greater than 8. In this paper we will consider the "dual" of this problem.

The set of points in the d-dimensional Euclidean space whose distance from a given line does not exceed a given positive number is called a cylinder. The given line is the cylinder's axis and the given number is its radius. The problem to be discussed is to find the maximum density $\rho(B^d)$ of unit balls in the d-dimensional Euclidean space touching a unit cylinder without overlapping each other.

Another natural motivation for the problem, which moreover has direct connection to the classical kissing number problem and the classical ball packing problem is the following. What is the maximum density of unit balls with disjoint interiors in the d-dimensional Euclidean space touching the Cartesian product of an i-dimensional affin space and a $(d-i)$-dimensional unit ball $(0 \leqslant i \leqslant d-1)$? Now the case $i=0$ is equivalent to the classical kissing number problem. On the other hand the case $i=d-1$ coincides with the classical ball packing problem in the $(d-1)$-dimensional space. For a detailed discussion about these problems see [4]. Here we consider the case $i=1$.

Let $C^d(r)$ denote the cylinder of radius r whose axis coincides with the x_d-axis of the Cartesian coordinate system. The (upper) density of a family of nonoverlapping unit balls touching $C^d(1)$ is defined by an appropriate limit (cf. [5], [12]) and is, roughly speaking, the portion of the volume of $C^d(3) \setminus C^d(1)$ which is occupied by the balls. Now, let $\rho(B^d)$ be the supremum of the upper densities of all such arrangements of unit balls. For technical reasons we will use a slightly different definition of $\rho(B^d)$ which is, of course, equivalent to the original one.

Let $C^d(r, 2h)$ denote the portion of $C^d(r)$ lying between the parallel hyperplanes of equation $x_d = -h$ and $x_d = h$. Furthermore, let $k_{2h}(B^d)$ be the maximum number of unit balls which can be arranged without overlapping in $C^d(3, 2h) \setminus C^d(1, 2h)$. Now, we define $\rho(B^d)$ by

$$\rho(B^d) = \limsup_{h \to \infty} \frac{k_{2h}(B^d)\, V(B^d)}{V(C^d(3, 2h) \setminus C^d(1, 2h))}.$$

Our problem was first discussed in [8] only in E^3, but in a more general form, replacing the unit balls with balls of radius r and looking for the analogous value $\rho(B^3(r))$. Surprisingly, the authors of [8] were able to determine the exact value of $\rho(B^3(r))$ for some special values of r.

In this paper we consider the problem in higher dimensions and we prove

Theorem 1. *For the density of nonoverlapping unit balls touching an infinite unit cylinder in E^d we have*

d	$\rho(B^d) \leqslant$	$\rho(B^d) \geqslant$
4	*0.4989*	*0.2718*
6	*0.1954*	*0.0811*
8	*0.0682*	*0.0247*

2. The Generalized Blichfeldt Method

Our estimates are based on the following generalization of the so-called Blichfeldt's enlargement method which was first proved in [7]. For the sake of completeness we include its easy proof (cf. [2], [7]).

Proposition 1. *Consider a family of unit balls and unit cylinders with mutually disjoint interiors in the d-dimensional Euclidean space. Leaving the centres fixed, replace the given geometrical balls with material balls all of radius $\sqrt{2}$. Let the density of the matter in each ball be equal to $2-r^2$ at a distance r from its centre. Similarly, leaving the axes fixed, replace the given geometrical cylinders with material cylinders all of radius $\sqrt{2}$. Let the density of the matter in each cylinder be equal to $2-r^2$ at a distance r from its axis. Then the total density at each point of E^d contributed by the enlarged balls and the enlarged cylinders containing the point is not greater than 2.*

Proof. Let P be an arbitrary point in E^d. By a suitable choice of the Cartesian coordinate system we may assume that P coincides with the origin. Suppose that k material balls $B_1^*, \ldots, B_k^*$ and l material cylinders $C_1^*, \ldots, C_l^*$ overlap at P. Let $(x_1^{(i)}, \ldots, x_d^{(i)}) \in E^d$ denote the centre of B_i^* for $1 \leqslant i \leqslant k$ and the orthogonal projection of P onto the axis of C_{i-k}^* for $k+1 \leqslant i \leqslant k+l$. Obviously, the distance between any two of these points is not smaller than 2, i.e.

$$\sum_{m=1}^{d} (x_m^{(i)} - x_m^{(j)})^2 \geqslant 4.$$

Summing up these inequalities for $1 \leqslant i < j \leqslant k+l$ we obtain

$$(k+l) \sum_{i=1}^{k+l} \sum_{j=1}^{d} \left(x_j^{(i)}\right)^2 - \sum_{j=1}^{d} \left(\sum_{i=1}^{k+l} x_j^{(i)} \right)^2 \geqslant 2(k+l)(k+l-1)$$

which implies

$$\sum_{i=1}^{k+l} \sum_{j=1}^{d} \left(x_j^{(i)}\right)^2 \geqslant 2(k+l-1).$$

Hence the total density at P is

$$\sum_{i=1}^{k+l} \left(2 - \sum_{j=1}^{d} \left(x_j^{(i)}\right)^2 \right) \leqslant 2(k+l) - 2(k+l-1) = 2.$$

This proves the proposition. □

Now, suppose that we have a family of nonoverlapping unit balls touching the cylinder $C^d(1)$. Replace the geometrical balls with material balls and the geometrical cylinder $C^d(1)$ with the material cylinder C^d_* as described in Proposition 1. Obviously, if we have $k_{2h}(B^d)$ geometrical balls in $C^d(3,2h)\setminus C^d(1,2h)$, then the corresponding material balls are contained in the cylinder $C^d(2+\sqrt{2},2h+2(\sqrt{2}-1))$. By Proposition 1, the total density of the enlarged balls and the enlarged cylinder in $C^d(2,2h+2(\sqrt{2}-1))$ is not greater than 2, i.e. the mass $M(C^d_*(2h+2(\sqrt{2}-1)))$ of the material cylinder and $k_{2h}(B^d)$ times the mass $M(C^d(2)\cap B^d_*)$ of the portion of a material ball lying in $C^d(2)$ divided by the volume of the cylinder $C^d(2,2h+2(\sqrt{2}-1))$ is smaller than or equal to 2, i.e.

$$\frac{M(C^d_*(2h+2(\sqrt{2}-1)))+k_{2h}(B^d)\,M(C^d(2)\cap B^d_*)}{V(C^d(2,2h+2(\sqrt{2}-1)))}\leqslant 2$$

from which we obtain an upper bound for $k_{2h}(B^d)$,

$$k_{2h}(B^d)\leqslant\frac{2\,V(C^d(2,2h+2(\sqrt{2}-1)))-M(C^d_*(2h+2(\sqrt{2}-1)))}{M(C^d(2)\cap B^d_*)}.$$

Now, the volume of $C^d(2,2h+2(\sqrt{2}-1))$ is

$$V(C^d(2,2h+2(\sqrt{2}-1)))=\kappa_{d-1}\,2^{d-1}\,(2h+2(\sqrt{2}-1))$$

where κ_d denotes the volume of the d-dimensional unit ball, i.e.

$$\kappa_d=\frac{\pi^{d/2}}{\Gamma(1+\frac{d}{2})}.$$

On the other hand, it is easy to see that

$$\begin{aligned}M(C^d_*(2h+2(\sqrt{2}-1)))&=(2h+2(\sqrt{2}-1))\int_0^{\sqrt{2}}\omega_{d-1}\,x^{d-2}\,(2-x^2)\,dx\\&=\frac{\kappa_{d-1}\,2^{(d+3)/2}}{d+1}\,(2h+2(\sqrt{2}-1))\end{aligned}$$

where $\omega_d=d\kappa_d$ is the surface area of the d-dimensional unit ball.

Substituting these formulas for the definition of density we obtain the following upper estimate for $\rho(B^d)$.

$$\begin{aligned}
\rho(B^d) &= \limsup_{h\to\infty} \frac{k_{2h}(B^d)\, V(B^d)}{V(C^d(3,2h)\setminus C^d(1,2h))} \\
&\leqslant \limsup_{h\to\infty} \frac{(2\,V(C^d(2,2h+2(\sqrt{2}-1))) - M(C^d_*(2h+2(\sqrt{2}-1))))\,V(B^d)}{V(C^d(3,2h)\setminus C^d(1,2h))\,M(C^d(2)\cap B^d_*)} \\
&= \limsup_{h\to\infty} \frac{\left(\kappa_{d-1}\,2^d\,(2h+2(\sqrt{2}-1)) - \dfrac{\kappa_{d-1}\,2^{(d+3)/2}}{d+1}\,(2h+2(\sqrt{2}-1))\right)\kappa_d}{\kappa_{d-1}\,(3^{d-1}-1)\,2h\,M(C^d(2)\cap B^d_*)} \\
&= \frac{\left(2^d - \dfrac{2^{(d+3)/2}}{d+1}\right)\kappa_d}{(3^{d-1}-1)\,M(C^d(2)\cap B^d_*)}.
\end{aligned}$$

It is still left to determine the mass $M(C^d(2)\cap B^d_*)$. For technical reasons we will consider the Cartesian coordinate system (different from the original one) in which the centre of B^d_* coincides with the origin of this coordinate system and the axis of $C^d(2)$ goes through the point $(2,0,\ldots,0)$ and is parallel to the x_d-axis. Simple computation shows that the intersection $C^d(2)\cap B^d_*$ lies between the hyperplanes of equation $x_d = -\sqrt{2}$ and $x_d = \sqrt{2}$. Let $-\sqrt{2} < c < \sqrt{2}$ and let us consider the $(d-1)$–dimensional balls $C^d(2)\cap H_c$ and $B^d_*\cap H_c$ where H_c is the hyperplane of equation $x_d = c$. It is easy to see that the intersection of these balls lies between the parallel hyperplanes of equation $x_1 = 0$ and $x_1 = \sqrt{2-c^2}$. Now, for $0 < a < \sqrt{2-c^2}$, let us consider the intersection of $C^d(2)\cap B^d_*\cap H_c$ and the hyperplane of equation $x_1 = a$. One can check immediately that this intersection is a $(d-2)$–dimensional ball of radius $\sqrt{-a^2+4a}$ for $a \leqslant -(c^2-2)/4$ and $\sqrt{2-a^2-c^2}$ for $a \geqslant -(c^2-2)/4$ in which the density of matter is equal to $2-a^2-b^2-c^2$ at a distance b from its centre. Taking these observations into account we obtain

$$\begin{aligned}
M(C^d(2)\cap B^d_*) = \int_{-\sqrt{2}}^{\sqrt{2}} \Bigg[&\int_0^{-\frac{x_d^2-2}{4}} \int_0^{\sqrt{-x_1^2+4x_1}} \omega_{d-2} x_2^{d-3}(2-x_1^2-x_2^2-x_d^2)\,dx_2\,dx_1 + \\
&\int_{-\frac{x_d^2-2}{4}}^{\sqrt{2-x_d^2}} \int_0^{\sqrt{2-x_1^2-x_d^2}} \omega_{d-2} x_2^{d-3}(2-x_1^2-x_2^2-x_d^2)\,dx_2\,dx_1 \Bigg]\,dx_d.
\end{aligned}$$

3. Numerical Results

Examining the formula for $M(C^d(2) \cap B_*^d)$, one can see immediately that in even dimensions it is easy to calculate this formula. We summarize the numerical data for dimensions 4, 6 and 8 in the next table. For the orientation of the reader we include the densities $\delta_L(B^d)$ of the densest lattice packing of unit balls (cf. [9], [1], [3]), too.

d	$M(C^d(2) \cap B_*^d)$	$\rho(B^d) \leqslant$	$\delta_L(B^d)$
4	5.2257	0.4989	0.6168
6	6.6402	0.1954	0.3729
8	6.8324	0.0682	0.2536

By a very simple construction one can give lower bounds for $\rho(B^d)$ as well. Let τ^d denote the kissing number of the d-dimensional unit ball, i.e. the maximum number of unit balls that can be simultaneously tangent to a further unit ball. Clearly, we can arrange τ^{d-1} unit balls such that they touch the cylinder $C^d(1)$ along the great sphere $x_1^2 + \cdots + x_{d-1}^2 = 1$ of $C^d(1)$. The iterates of the translation $(0, \ldots, 0, 2)$ and their inverses generate a sequence of such 'rings' of balls with disjoint interiors. Calculating the density of this arrangement we obtain lower bounds for $\rho(B^d)$,

$$\begin{aligned} \rho(B^d) &\geqslant \lim_{h \to \infty} \frac{h\tau^{d-1} V(B^d)}{V(C^d(3, 2h) \setminus C^d(1, 2h))} \\ &= \frac{h\tau^{d-1}\kappa_d}{2h\kappa_{d-1}(3^{d-1} - 1)} \\ &= \frac{\kappa_d}{2\kappa_{d-1}} \frac{\tau^{d-1}}{3^{d-1} - 1}. \end{aligned}$$

The numerical data, using the constructive lower estimates for τ^d in [11], are

d	$\tau^{d-1} \geqslant$	$\rho(B^d) \geqslant$
4	12	0.2718
6	40	0.0811
8	126	0.0247

Finally, we note that our method does not give satisfactory results in the 3-dimensional case. Indeed, the obtained upper bound is greater than $\pi/\sqrt{18}$, the density of the densest lattice packing of unit balls (cf. [6]).

$M(C^3(2) \cap B_*^3)$	$\rho(B^3) \leqslant$	$\delta_L(B^3)$
4.2112	0.7459	0.7404

For comparison, a certain family of unit balls in E^3 is of density 0.5843... and we conjecture it to be a densest one. The authors of [8] were able to prove that $\rho(B^3) \leqslant 0.5846$.

References

[1] Blichfeldt, H.F.: On the minimum value of positive real quadratic forms in 6 variables, *Bull. Amer. Math. Soc.* **31** (1925), 386.

[2] Blichfeldt, H.F.: The minimum value of quadratic forms, and the closest packing of spheres, *Math. Ann.* **101** (1929), 605-608.

[3] Blichfeldt, H.F.: The minimum value of positive quadratic forms in six, seven and eight variables, *Math. Z.* **39** (1934), 1-15.

[4] Conway, J.H., Sloane, N.J.A.: *Sphere packings, lattices and groups*, Springer Verlag, New York, 1988.

[5] Fejes Tóth, L.: *Lagerungen in der Ebene, auf der Kugel und im Raum*, zweite Auflage, Springer Verlag, Berlin, 1972.

[6] Gauss, C.F.: Untersuchungen über die Eigenschaften der positiven ternären quadratischen Formen von Ludwig August Seber, *J. Reine Angew. Math.* **20** (1840), 312-320.

[7] Heppes, A., Szabó, L.: On the number of cylinders touching a ball, *Geom. Dedicata* **40** (1991), 111-116.

[8] Horváth, J., Temesvári, Á.H., Závoti, J.: Über Kugelpackungen herum einen Zylinder, Manuscript.

[9] Korkin, A., Zolotarev, G.: Sur les formes quadratiques positives quaternaires, *Math. Ann.* **5** (1872), 581-583.

[10] Kuperberg, W.: Problem 3.3, in: *DIMACS Workshop on Polytopes and Convex Sets*, Rutgers University, January 10, 1990.

[11] Leech, J., Sloane, N.J.A.: Sphere packings and error-correcting codes, *Canad. J. Math.* **23** (1971), 718-745.

[12] Rogers, C.A.: *Packing and covering*, Cambridge University Press, Cambridge, 1964.