21-neighbour packings of equal balls in the 4-dimensional space

László Szabó
Eötvös Loránd University
Department of Geometry
Rákóczi út 5
H-1088 Budapest
Hungary

August 2, 2000

Abstract

A packing of equal balls in the n-dimensional Euclidean space is said to be a k-neighbour packing if each ball is touched by at least k others. We show that any 21-neighbour packing of congruent balls in the 4-dimensional space has positive density and there exist 18-neighbour packings with 0 density.

1. Introduction

Let $\mathcal{F}$ be a family of closed sets in the n-dimensional Euclidean space. If the interiors of the members of $\mathcal{F}$ are disjoint then $\mathcal{F}$ is said to be a packing. Two members of the packing $\mathcal{F}$ are called neighbours if their boundaries have a point in common. A packing in which each set has at least k neighbours is called a k-neighbour packing. Following the terminology of L. Fejes Tóth, the maximum number of neighbours of a closed convex set S in all packings of congruent copies of S is called the Newton number of S. For some interesting results and conjectures about k-neighbour packings and Newton numbers, see [3]. Here we recall only those which concern packings of congruent balls.

Let B^n denote the n-dimensional unit ball and $N(B^n)$ the Newton number of B^n. It is well known among geometers that the problem of finding $N(B^3)$ was once a point of controversy between Newton and Gregory: Can a rigid material ball be brought into contact with 13 other balls of the same size? It took almost two hundred years before Hoppe proved that Newton's guess, the 'no', is correct [7]. After this it should not be surprising that today $N(B^n)$ is only known for two further dimensions, $N(B^8) = 240$ and $N(B^{24}) = 196560$ [8].

In discrete geometry it is natural to look for properties such that the members of a packing with this property occupy a positive percentage of the whole space, i.e. the packing has positive density (for a precise definition of the density, see [9]). Immediate intuition says that a k-neighbour packing of congruent balls in the n-dimensional Euclidean space must have positive density if k is nit much smaller than $N(B^n)$.

This intuitive statement is confirmed in 2 and 3 dimensions. L. Fejes Tóth proved in [6] that the thinnest 5-neighbour packing of congruent circles in the plane has density $\frac{\sqrt{3}\pi}{7}$. On the other hand, as Figure 1 shows, there are 4-neighbour packings with 0 density.

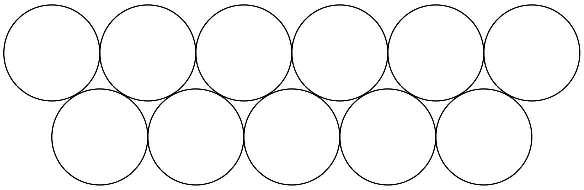

Figure 1.

The 3-dimensional case seemed to be more complicated. In [1] and [2] it is proved that any 10-neighbour packing of congruent balls has positive density. Let us note that the thinnest packing of this type has not yet been determined. The following example shows that one cannot expect a similar result by replacing 10 with a smaller integer. Consider a layer of balls such that the centres lie on the same plane and each ball is touched by six others. One can arrange two layers of this type in parallel position so that their union is a 9-neighbour packing and clearly has 0 density.

In this paper we prove

Theorem 1. *Any 21-neighbour packing of congruent balls in the 4-dimensional space has positive density.*

As the following construction shows, there exist 18-neighbour packings with 0 density. Let $\mathcal{C} = \{\mathbf{x} \in \mathbb{E}^4 \mid \mathbf{x} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 + \lambda_4\mathbf{v}_4,\ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z},\ \lambda_4 = 0, 1\}$ where $\mathbf{v}_1 = (2,0,0,0)$, $\mathbf{v}_2 = (0,2,0,0)$, $\mathbf{v}_3 = (1,1,1,1)$, $\mathbf{v}_4 = (0,0,2,0)$. The unit balls with centres in $\mathcal{C}$ form a 18-neighbour packing and, obviously, has 0 density.

2. An observation

The following simple argument brings more light to the n-dimensional problem discussed in the introduction.

Let σ_n denote the largest integer σ such that σ rigid balls congruent with B^n can be brought into contact with B^n so that all contact points lie on a fixed closed hemisphere of B^n. Now we show that all $(\sigma_n + 1)$-neighbour packings of congruent balls in the n-dimensional Euclidean space have positive density.

Let $\mathcal{F}$ be a $(\sigma_n + 1)$-neighbour packing of balls congruent with B^n. Consider the Dirichlet tessallation associated with $\mathcal{F}$ (see [5]). Suppose that we can choose points P_i, $(i = 1, 2, \dots)$, from the interiors of the Dirichlet cells so that the distances $\operatorname{dist}(O_i, P_i)$, $(i = 1, 2, \dots)$, tend to infinity where O_i denotes the centre of the ball of $\mathcal{F}$ which is contained in the cell of P_i. As $\mathcal{F}$ is a $(\sigma_n + 1)$-neighbour packing, the surface of the unit ball centered at O_i contains at least $(\sigma_n + 1)$ contact points. Obviously, non of these points is visible from P_i. Let us say they lie in the shadow corresponding to P_i. As the distances $\operatorname{dist}(O_i, P_i)$, $(i = 1, 2, \dots)$, tend to infinity, the corresponding shadows approximate the hemisphere. A standard calculus argument shows that there is a convergent subsequence of the sequence of $(\sigma_n + 1)$-tuples of contact points. The limit configuration is contained in a hemisphere, a contradiction. Thus there is a positive constant R such that each Dirichet cell is contained in a ball of radius R which implies that the density within each cell is at least $\frac{1}{R^n}$.

3. Proof of Theorem 1

First we note that if in a packing of unit balls the ball B^n touches two others then the spherical distance of the contact points on B^n is at least $\frac{\pi}{3}$. Now, using the above observation it is sufficient to show

Proposition 1. *In the 4-dimensional space it is impossible to find 21 points on a closed hemisphere with pairwise spherical distances at least $\frac{\pi}{3}$.*

Throughout the proof we use the following notations.

$$\begin{aligned}
S &= \{(v,w,x,y) \in \mathbb{E}^4 \mid v^2 + w^2 + x^2 + y^2 = 1\},\\
H &= \{(v,w,x,y) \in \mathbb{E}^4 \mid v^2 + w^2 + x^2 + y^2 = 1,\ 0 \le y\},\\
A &= \{(v,w,x,y) \in \mathbb{E}^4 \mid v^2 + w^2 + x^2 + y^2 = 1,\ 0 \le y \le \tfrac{1}{2}\},\\
E &= \{(v,w,x,0) \in \mathbb{E}^4 \mid v^2 + w^2 + x^2 = 1\},\\
N &= (0,0,0,1).
\end{aligned}$$

Finally, if the points P and Q belongs to S then $\mathrm{dist}(P, Q)$ denotes the spherical distance between them.

First we prove

Lemma 1. *The set A cannot contain 17 points with pairwise spherical distances at least $\frac{\pi}{3}$.*

Proof. Suppose that the set A contains 17 points $P_1, \ldots, P_{17}$ with pairwise spherical distances at least $\frac{\pi}{3}$. For $1 \leq i \leq 17$, let Q_i denote the image of the point P_i under the spherical orthogonal projection from N onto E, i.e. the nearest point of E belonging to the (unique) great circle containing N and P_i. Note that the points $Q_1, \ldots, Q_{17}$ are distinct. Let $1 \leq i < j \leq 17$ and let us consider the sphere $V \cap S$ where V is the linear subspace of the 4-dimensional space spanned by Q_i, Q_j, N. Obviously, $\mathrm{dist}(P_i, Q_i) \leq \frac{\pi}{3}$ and $\mathrm{dist}(P_j, Q_j) \leq \frac{\pi}{3}$. Furthermore, it is easy to see that if $\mathrm{dist}(Q_i, Q_j) < \arccos \frac{1}{\sqrt{3}}$ then $\mathrm{dist}(P_i, P_j) < \frac{\pi}{3}$ (see Figure 2).

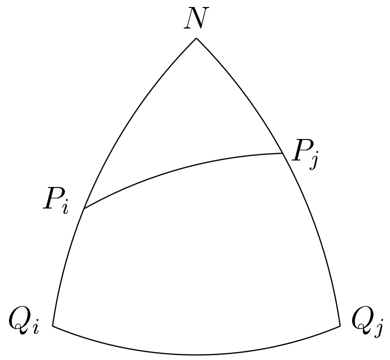

Figure 2.

Hence we conclude that there exist 17 points on E with pairwise spherical distances at least $\arccos \frac{1}{\sqrt{3}}$.

Finally we recall a result of L. Fejes Tóth concerning the Tammes Problem [4]. Assume we have m points on a unit sphere with pairwise spherical distances at least a then the inequality $(2m-4)\triangle \leq 4\pi$ holds where $\triangle$ denotes the area of an equilateral spherical triangle of side a. Now, for $m = 17$ and $a = \arccos \frac{1}{\sqrt{3}}$, easy computation shows that $(2m-4)\triangle > 4.2\pi$, a contradiction. $\square$

Turning back to the proof of Proposition 1 suppose that H contains 21 points $P_1, \ldots, P_{21}$ with pairwise spherical distances at least $\frac{\pi}{3}$. Assume that these points are labelled so that $P_1, \ldots, P_t$ lie in $H \setminus A$ while the rest $q = 21 - t$ points lie in A. Let us consider the set $\mathcal{P} = \{P_1, \ldots, P_{21}, R_1, \ldots, R_t\}$ where R_j is the image of P_j under the reflection at the equator E, $(1 \leq j \leq t)$. It is easy to see that the spherical distance of any two points of $\mathcal{P}$ is at least $\frac{\pi}{3}$. In view of Lemma 1 we have $q < 17$ and therefore $|\mathcal{P}| = 2t + q = 42 - q > 25$. But it is well known that $N(B^4) \leq 25$ [8], a contradiction.

4. Concluding remarks

In connection with Proposition 1 note that the following 18 points on the hemisphere $\{(v, w, x, y) \in \mathbb{E}^4 \mid v^2 + w^2 + x^2 + y^2 = 1,\ x - y \geq 0\}$ form a system with pairwise spherical distances at least $\frac{\pi}{3}$.

$$\begin{aligned}
&(\pm 1, 0, 0, 0),\\
&(0, \pm 1, 0, 0),\\
&(0, 0, 1, 0),\\
&(0, 0, 0, -1),\\
&(\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \tfrac{1}{2}, \pm\tfrac{1}{2}),\\
&(\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}).
\end{aligned}$$

We do not know if such a system of 19 or 20 points exists.

References

[1] Bezdek, A., Bezdek, K.: A note on ten-neighbour packings of equal balls, *Beiträge Algebra Geom.* **27** (1988), 49-53.

[2] Fejes Tóth, G.: Ten-neighbour packing of equal balls, *Period. Math. Hungar.* **12/2** (1981), 125-127.

[3] Fejes Tóth, G.: New results in the theory of packing and covering, in: *Convexity and its applications*, eds. Gruber, P.M., Wills, J.M., Birkhäuser Verlag, 1983, pp. 318-359.

[4] Fejes Tóth, L.: On the densest packing of spherical caps, *Amer. Math. Monthly* **56** (1949), 330-331.

[5] Fejes Tóth, L.: *Lagerungen in der Ebene, auf der Kugel und im Raum*, zweite Auflage, Springer Verlag, 1972.

[6] Fejes Tóth, L.: Five-neighbour packing of convec discs, *Period. Math. Hungar.* **4** (1973), 221-229.

[7] Hoppe, R.: Bemerkungen der Redaktion zu 'Bender, C., Bestimmung der größten Anzahl gleich grosser Kugeln, welche sich auf eine Kugel von demselben Radius wie die übrigen auflegen lassen, *Arch. Math. Phys.* **56** (1874), 302-306.' *Ibid.* 307-312.

[8] Odlyzko, A.M., Sloane, N.J.A.: New bounds on the number of unit spheres that can touch a unit sphere in n-dimensions, *J. Combin. Theory Ser. A* **26** (1979), 210-214.

[9] Rogers, C.A.: *Packing and covering*, Cambridge University Press, 1964.