Classification Results for Non-Mixed and Mixed Optimal Covering Codes: a Survey

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1. Introduction and Notation

n-tuple : n-dimensional vector, whose coordinates are symbols from a finite alphabet such as $\{1, 2\}$ or $\{0, 1\}$ or $\{Y, N\}$ or $\{\bullet, \bullet\}$ or $\{1, 2, 3\}$ or $\{0, 1, 2\}$ or $\{1, 2, x\}$ etc.

Code of length n : a collection of n-tuples.

Hamming space : the collection of all n-tuples.

Hamming distance of two n-tuples : the number of coordinates where the corresponding symbols are different.

Sphere in the Hamming space : the set of such n-tuples that are within a given distance /- the radius -/ of a given n-tuple /- the center -/.

Covering code of length n with a given radius : a collection of n-tuples so that the union of the corresponding spheres entirely cover the Hamming space.

Optimal covering code of length n, with covering radius R : a covering code of least possible cardinality for the given parameters n, R.

Notation for the size of optimal covering codes : $K_q(n, R)$

Remark: Very few exact values of $K_q(n, R)$ are known!

Description of codes:

a) Exact (scientific) form

$$
\begin{array}{ll}
\{ & (c_{11}, c_{12}, \ldots, c_{1n}), \\
 & (c_{21}, c_{22}, \ldots, c_{2n}), \\
 & \ldots\ldots\ldots\ldots\ldots\ldots \\
 & (c_{M1}, c_{M2}, \ldots, c_{Mn})\}
\end{array}
$$

or

$$
\begin{array}{rcl}
c_1 & = & (c_{11}, c_{12}, \ldots, c_{1n}), \\
c_2 & = & (c_{21}, c_{22}, \ldots, c_{2n}), \\
. & . & \ldots\ldots\ldots\ldots\ldots\ldots \\
c_M & = & (c_{M1}, c_{M2}, \ldots, c_{Mn}).
\end{array}
$$

Description of codes (continued):

b) Arrangement of the codewords into a matrix

$$\begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ . & . & \dots & . \\ c_{M1} & c_{M2} & \dots & c_{Mn} \end{pmatrix}$$

c) Arrangement of the codewords into an array or file

$$\begin{matrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ . & . & \dots & . \\ c_{M1} & c_{M2} & \dots & c_{Mn} \end{matrix}$$

Example for spheres

$$C = \{(0,1,0),(1,0,1)\}$$

Sphere at center c_1 (with radius $R = 1$):

$$\{(0,0,0),(0,1,0),(0,1,1),(1,1,0)\}$$

Sphere at center c_2 (with radius $R = 1$):

$$\{(0,0,1),(1,0,0),(1,0,1),(1,1,1)\}$$

The given example shows that $K_2(3,1) = 2$.

2. Football pools

An interesting application of covering codes is trying to win at football pools (football toto). Filling and submitting a football pool ticket is to give forecasts for the outcome of a given number football matches and to win a prize in the hope that the forecasts will prove to be completely or nearly correct.

Each football match has three possible outcomes: home win '1', draw 'x' or away win '2'. The gambler may win first prize, second prize, third prize (or eventually fourth prize) with his forecasts. To win first prize all forecasts have to be correct. To win second prize one of the forecasts may be incorrect.

If a football pool gambler wishes to guarantee the first prize for himself, clearly, he has to submit 3^n tickets.

The football pool problem: How many tickets are to be submitted to guarantee the second prize? How many for the third prize? Fourth prize? (And so on.)

3. Numerical examples for $q = 3$

a) $n = 3, R = 1$; b) $n = 5, R = 2$; c) $n = 4, R = 2$.

1	1	1
1	2	2
2	1	2
2	2	1
x	x	x

0	0	0	0	0
0	0	0	1	1
1	1	2	2	2
1	2	1	2	2
1	2	2	0	0
2	1	1	0	1
2	1	1	1	0
2	2	2	2	2

1	1	1	1
2	2	2	2
3	3	3	3

d) $n = 6, R = 3$. (There are many inequivalent solutions!)

0	0	0	0	0	0
0	0	1	1	1	1
1	1	0	0	1	2
1	1	2	2	2	0
2	2	1	2	0	1
2	2	2	1	2	2

0	0	0	0	0	0
0	0	0	1	1	1
1	1	1	0	0	1
1	1	1	2	2	2
2	2	2	1	1	2
2	2	2	2	2	0

0	0	0	0	0	0
0	0	0	0	1	1
0	0	0	0	2	2
1	1	1	1	0	1
1	1	1	1	1	0
2	2	2	2	2	2

0	0	0	0	0	0
0	0	0	0	1	1
0	0	0	1	0	1
1	1	1	0	0	1
1	1	1	1	1	0
2	2	2	2	2	2

0	0	0	0	0	0
0	0	0	0	1	1
1	1	1	1	0	0
1	1	1	1	2	2
2	2	2	2	1	1
2	2	2	2	2	2

0	0	0	0	0	0
0	0	0	0	1	1
1	1	1	1	0	0
1	1	1	1	2	2
2	2	2	2	1	2
2	2	2	2	2	1

4. Classification of codes

equivalent (isomorphic) codes : rearranging the coordinates, renaming the symbols of the alphabet results in equivalent codes.

Example

1 1 1	1 1 1	1 1 0
1 2 2	2 2 1	2 2 0
2 1 2	1 2 2	1 2 2
2 2 1	2 1 2	2 1 2
x x x	0 0 0	0 0 1

are all equivalent codes.

Classification results are useful in any cases when we are interested in finding different structures for the same covering problem.

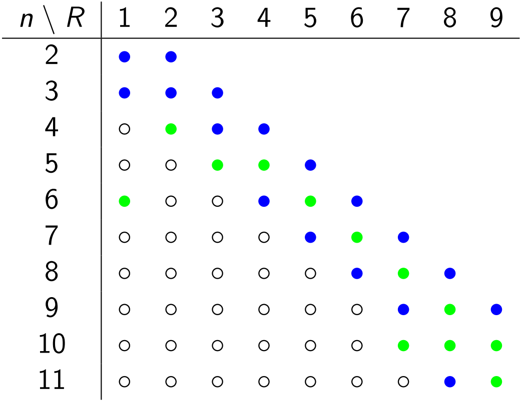
5. Solved and unsolved cases

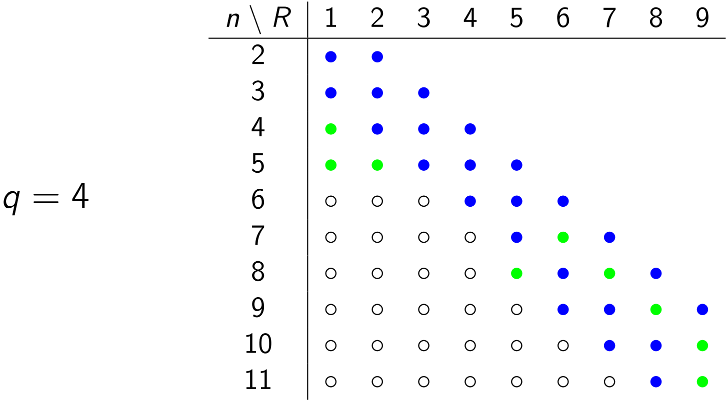
- ● = solved and classified (blue)
- ● = solved but not yet classified (green)
- ○ = unsolved

$q = 6$

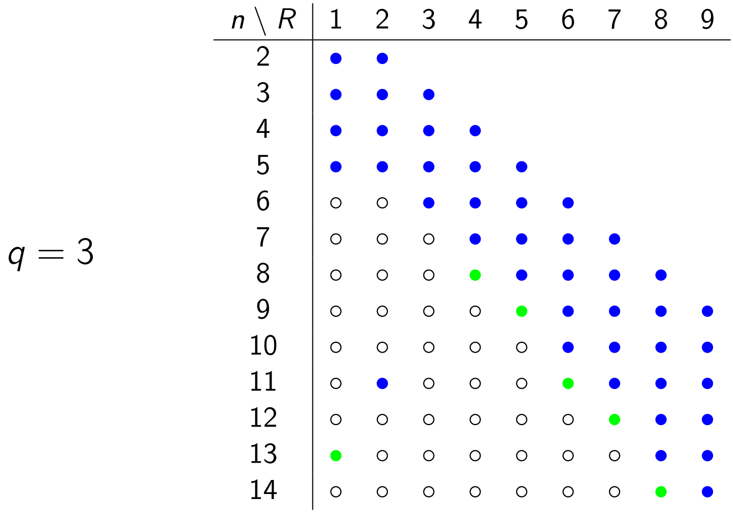
$n \setminus R$	1	2	3	4	5	6	7	8	9
2	● (blue)	● (blue)							
3	● (green)	● (green)	● (blue)						
4	● (green)	● (green)	● (green)	● (blue)					
5	○	○	● (green)	● (green)	● (blue)				
6	○	○	○	● (green)	● (green)	● (blue)			
7	○	○	○	○	● (blue)	● (green)	● (blue)		
8	○	○	○	○	○	● (blue)	● (green)	● (blue)	
9	○	○	○	○	○	○	● (green)	● (green)	● (blue)
10	○	○	○	○	○	○	○	● (green)	● (green)

$q = 5$

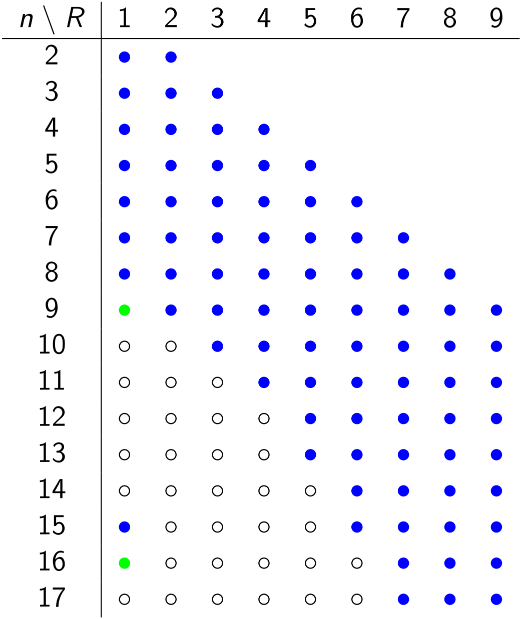




$q = 4$
$n \setminus R$
1 2 3 4 5 6 7 8 9
2
3
4
5
6
7
8
9
10
11


q = 3
n \ R
1 2 3 4 5 6 7 8 9
2
3
4
5
6
7
8
9
10
11
12
13
14

$q = 2$



$q = 2$ (continued)

$n \setminus R$	1	2	3	4	5	6	7	8	9
18	○	○	○	○	○	○	○	● (blue)	● (blue)
19	○	○	○	○	○	○	○	● (blue)	● (blue)
20	○	○	○	○	○	○	○	○	● (blue)
21	○	○	○	○	○	○	○	○	● (blue)
22	○	○	○	○	○	○	○	○	○
23	○	○	● (blue)	○	○	○	○	○	○
24	○	○	○	○	○	○	○	○	○
25	○	○	○	○	○	○	○	○	○
26	○	○	○	○	○	○	○	○	○
27	○	○	○	○	○	○	○	○	○
28	○	○	○	○	○	○	○	○	○
29	○	○	○	○	○	○	○	○	○
30	○	○	○	○	○	○	○	○	○
31	● (green)	○	○	○	○	○	○	○	○
32	● (green)	○	○	○	○	○	○	○	○
33	○	○	○	○	○	○	○	○	○

6. The chronology of nontrivial known classification results

Zaremba (1952) : Uniqueness for $K_2(7, 1) = 16$.

Kalbfleisch and Stanton (1968, 1969) : For $K_2(4, 1) = 4$, there are 2 inequivalent solutions. – Uniqueness for $K_2(5, 1) = 7$ and $K_3(4, 1) = 9$.

Delsarte and Goethals (1975) : Uniqueness for $K_2(23, 3) = 4096$ and $K_3(11, 2) = 729$.

Kolev (1993) : For $K_{3,2}(1, 4, 1) = 8$, there are 2 inequivalent solutions.

Östergård (1996) : Uniqueness for $K_3(3, 1) = 5$.

Östergård and Weakley (2000) : For $K_2(2R+2, R) = 4$, there are $\lfloor (\frac{R}{2}+1)^2 \rfloor$ inequivalent solutions. For $K_2(6,1) = 12$, there are 2 inequivalent solutions. For $K_2(8,1) = 32$, there are 10 inequivalent solutions. For $K_2(7,2) = 7$, there are 3 inequivalent solutions. For $K_2(8,2) = 12$, there are 277 inequivalent solutions.

The same authors (2002) : For $K_3(5,1) = 27$, there are 17 inequivalent solutions.

Kéri and Östergård (2003) : For $K_4(4,2) = 7$, there are 8 inequivalent solutions.

Bertolo, Östergård and Weakley (2004) : For $K_2(9,2) = 16$, there are 4 inequivalent solutions. For $K_3(5,2) = 8$, uniqueness is proved.

Kaski and Östergård (2006) : For $K_q(2,1) = q$, the number of inequivalent optimal covering codes agrees with the number of multisets of positive integers whose sum is q. For $K_2(9,3) = 7$, there are 8 inequivalent solutions.

Kéri and Östergård (2006) : For $K_2(2R+3, R) = 7$, the the number of inequivalent optimal covering codes is the coefficient of x^{R-1} in the expansion of

$$\frac{1}{(1-x)^3(1-x^2)^2(1-x^3)} = 1 + 3x + 8x^2 + 17x^3 + 33x^4 + \ldots.$$

The same authors (2007) : For $K_{3,2}(3,1,1) = 9$, there are 4 inequivalent solutions. For $K_{3,2}(6,1,3) = 9$, there are also 4 inequivalent solutions.

The same authors (2008) : For $K_2(10,3) = 12$, there are 11481 inequivalent solutions.

$K_2(2R+3, R) = 7$ ... coefficient of x^{R-1} in ...

$$\frac{1}{(1-x)^3(1-x^2)^2(1-x^3)} = 1 + 3x + 8x^2 + 17x^3 + 33x^4 + \dots$$

n	$R = 1$	$R = 2$	$R = 3$
2	$\mathbf{2}^2$	$\mathbf{1}^1$	
3	$\mathbf{2}^1$	$\mathbf{2}^3$	$\mathbf{1}^1$
4	b $\mathbf{4}^2$	$\mathbf{2}^2$	$\mathbf{2}^4$
5	b $\mathbf{7}^1$ b	$\mathbf{2}^1$	$\mathbf{2}^3$
6	c $\mathbf{12}^2$ c	g $\mathbf{4}^4$	$\mathbf{2}^2$
7	h $\mathbf{16}^1$ h	g $\mathbf{7}^3$ g	$\mathbf{2}^1$
8	c $\mathbf{32}^{10}$	z $\mathbf{12}^{277}$ g	g $\mathbf{4}^6$
9	o 62 j	p $\mathbf{16}^4$ y	g $\mathbf{7}^8$ g

Östergård and Pottonen (2009) : For $K_2(15,1) = 2048$, there are 5983 inequivalent solutions.

New, till now unpublished, computational results :

For any cases where $K_3(n,R) = 3$, the the number of inequivalent optimal covering codes is the coefficient of $x^{3R-2n+2}$ in the expansion of

$$\frac{1}{(1-x)^3(1-x^2)(1-x^3)} = 1 + 3x + 7x^2 + 14x^3 + 25x^4 + \dots .$$

For $K_3(6,3) = 6$, there are 28 inequivalent solutions.

Uniqueness is proved for $K_4(3,1) = 8$ and for $K_5(3,1) = 13$.

A lot of new, computational classification results are found for mixed ternary/binary and other mixed covering codes, e.g., uniqueness is proved for $K_{3,2}(2,2,1) = 6$ and for $K_{3,2}(2,4,1) = 20$.

$K_3(n,R) = 3$ ... coefficient of $x^{3R-2n+2}$ in ...

$$\frac{1}{(1-x)^3(1-x^2)(1-x^3)} = 1 + 3x + 7x^2 + 14x^3 + 25x^4 + \dots$$

n	$R=4$	$R=5$	$R=6$	$R=7$	$R=8$
5	$\mathbf{3}^{25}$	$\mathbf{1}^{1}$			
6	$\mathbf{3}^{7}$	$\mathbf{3}^{41}$	$\mathbf{1}^{1}$		
7	$\mathbf{3}^{1}$	$\mathbf{3}^{14}$	$\mathbf{3}^{64}$	$\mathbf{1}^{1}$	
8	q 9	$\mathbf{3}^{3}$	$\mathbf{3}^{25}$	$\mathbf{3}^{95}$	$\mathbf{1}^{1}$
9	u 11–18 m	p 6 m	$\mathbf{3}^{7}$	$\mathbf{3}^{41}$	$\mathbf{3}^{136}$
10	m 17–36 m	q 9–12 m	$\mathbf{3}^{1}$	$\mathbf{3}^{14}$	$\mathbf{3}^{64}$
11	m 30–81 m	m 11–27 m	q 9	$\mathbf{3}^{3}$	$\mathbf{3}^{25}$
12	y 62–175 x	y 18–54 m	u 10–18 m	d 6 m	$\mathbf{3}^{7}$
13	y 123–340 y	m 33–108 o	m 13–36 q	q 9–12 m	$\mathbf{3}^{1}$

Thank you for your attention.