Research visit of Patrick Govaerts at MTA SZTAKI

Report on the stay from 2 November 2003 till 30 January 2004

Contents

1 Personalia

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	Ghent University, 25 April 2003
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2 Research objectives

The aim of this research visit is the study of objects in Galois geometries that are closely related on the one hand to Latin squares and on the other hand to optimal linear codes.

Latin squares

A *Latin square of order s* is an $s \times s$ array with entries in the set $S = \{1, 2, \ldots s\}$ such that each row and each column contains each element of S exactly once. Two Latin squares $L = (l_{ij})$ and $M = (m_{ij})$ of order s are called *orthogonal* if the set $\{(l_{ij}, m_{ij}) : 1 \leq i, j \leq s\}$ equals the set S^2. A set of t mutually orthogonal Latin squares of order s is denoted by t MOLS(s). It is called *maximal* if there exists no Latin square of order s that is orthogonal to all its elements, in which case it is denoted by t MAXMOLS(s).

Latin squares have many applications, for example in the design of statistical experiments, in the organisation of tournaments and in the domain of massive parallel computing. The problem of determining the numbers t and s for which t MOLS(s) exist is an old one, originating by Euler [11] in 1782.

Optimal linear codes

In many cases it is inevitable that errors occur during the transmission of data or that stored data is corrupted. In the former case think of atmospheric disturbances that distort data transmitted by a satellite; in the latter case of scratches on a Compact Disc. By adding a certain amount of redundant information to the transmitted or stored data, the theory of error-correcting codes can overcome the possibly disastrous effects of such inevitable errors by allowing to retrieve correct information form corrupted data.

For practical purposes, there is great interest in linear codes. A *linear* $[n, k, d; q]$*-code* is a k-dimensional vector space in $V(n, q)$, the n-dimensional vector space over the finite field of order q, with minimum Hamming distance d. The parameters of the code are the *length n* (which determines the size of one codeword, i.e., the amount of data needed to encode one piece of information), the *dimension d* (which determines the number of codewords, i.e., the number of different messages that can be encoded) and the *minimum distance d* (which determines the number of errors the receiver of a message can detect and correct). Hence one is interested in codes with small length, large dimension and large minimum distance. However, these different requirements contradict one another: there exist theoretical limits on the size of one of these parameters in function of the other two. Codes that attain one of these limits are called *optimal*. Here we mention two of these bounds. The *Griesmer bound* [14] states that an $[n,k,d;q]$-code satisfies $n \geq \sum_{i=0}^{k-1} \lceil d/q^i \rceil$, while the *Singleton bound* [19] for linear codes gives the restriction $d \leq n-k+1$. A code attaining the former bound is called a *code attaining the Griesmer bound*; one attaining the latter bound is called an *MDS code*.

Latin squares and partial spreads

One way to construct sets of mutually orthogonal Latin squares is to find (s,r)-nets[1], since it is well-known that a set of $r-2$ MOLS(s) yields an (s,r)-net and vice versa. It is here that partial spreads[2] come into play, for from a partial spread an (s,r)-net can be obtained. If one wants to construct maximal sets of MOLS from partial spreads, then the partial spread must be maximal, but this necessary condition is not sufficient. In addition, the net that originates from the partial spread must also be maximal. It is this part of the construction, showing that the associated net is maximal, that is often the most difficult. Still, in several instances, see e.g. [1] and [12], this obstacle was overcome to show that certain maximal partial spreads yield sets of MAXMOLS.

Optimal codes, arcs and blocking sets

To clarify the relation between optimal linear codes and certain sets of points in finite projective spaces, we mention the following two equivalences[3]:

- there is a one-one correspondence between certain types of multiple blocking sets in PG(n,q), so-called $\{f,m;n,q\}$-minihypers where f and m satisfy well-defined conditions, and linear codes meeting the Griesmer bound;
- the study of MDS codes is equivalent to the study of arcs in finite projective spaces.

3 Results obtained during the visit

We will use the following notations and definitions.

- GF(q) denotes the finite field of order q. Hence $q = p^h$ for some prime p and some positive integer h.

[1]For the definition of (s,r)-nets, see Subsection 3.4.
[2]For the definition of partial spreads, see the beginning of Section 3.
[3]For the definitions of multiple blocking sets, minihypers and arcs, see the beginning of Section 3.

- $\mathrm{AG}(n,q)$ denotes the n-dimensional affine space over $\mathrm{GF}(q)$.
- $\mathrm{PG}(n,q)$ denotes the n-dimensional projective space over $\mathrm{GF}(q)$.
- A *t-space* in $\mathrm{PG}(n,q)$ is a t-dimensional subspace of $\mathrm{PG}(n,q)$.
- A *point* (respectively *line, plane, solid, hyperplane*) in $\mathrm{PG}(n,q)$ is a t-space where $t=0$ (respectively $t=1$, $t=2$, $t=3$, $t=n-1$).
- A *k-fold blocking set with respect to t-spaces in* $\mathrm{PG}(n,q)$ is a set of points in $\mathrm{PG}(n,q)$ that intersects every t-space of $\mathrm{PG}(n,q)$ in at least k points.
- An $\{f,m;n,q\}$-*minihyper* is an m-fold blocking set with respect to hyperplanes in $\mathrm{PG}(n,q)$ that is not an $(m+1)$-fold blocking set with respect to hyperplanes in $\mathrm{PG}(n,q)$.
- A *k-arc* in $\mathrm{PG}(n,q)$ is a set of k points such that no hyperplane contains $n+1$ of them.
- A *partial t-spread* in $\mathrm{PG}(n,q)$ is a set of mutually disjoint t-spaces in $\mathrm{PG}(n,q)$.

3.1 Blocking sets in $\mathrm{PG}(n,2)$

As explained above, a *blocking set with respect to t-spaces in* $\mathrm{PG}(n,q)$ is a set of points that has nonempty intersection with every t-space of $\mathrm{PG}(n,q)$. It is called *minimal* if deleting any point from the set results in a set that no longer blocks all t-spaces. A blocking set with respect to lines in a projective plane is simply called a *blocking set*. The smallest blocking sets with respect to t-spaces are characterised in the following theorem.

Theorem 3.1 (Bose and Burton [7]) *If B is a blocking set with respect to t-spaces in* $\mathrm{PG}(n,q)$, *then* $|B| \geq |\mathrm{PG}(n-t,q)|$. *Equality holds if and only if B is an $(n-t)$-space.*

A blocking set with respect to t-spaces that contains an $(n-t)$-space is called *trivial*. The smallest nontrivial blocking sets with respect to t-spaces in $\mathrm{PG}(n,q)$ are characterised for $q>2$ in Theorem 3.2.

Theorem 3.2 (Beutelspacher [3], Heim [15]) *In* $\mathrm{PG}(n,q)$, $q>2$, *the smallest nontrivial blocking sets with respect to t-spaces,* $1 \leq t \leq n-1$, *are cones with vertex an $(n-t-2)$-space π_{n-t-2} and base a nontrivial blocking set of minimal cardinality in a plane skew to π_{n-t-2}.*

It is known that if $q>2$, then $\mathrm{PG}(2,q)$ has a nontrivial blocking set and that the size of such a nontrivial blocking set is substantially bigger than $q+1$, the size of a line.

Theorem 3.3 *Let B be a nontrivial blocking set of* $\mathrm{PG}(2,q)$, $q>2$.

1. **(Bruen [8])** $|B| \geq q+\sqrt{q}+1$, *with equality if and only if B is a Baer subplane.*
2. **(Blokhuis [4])** *If q is a prime, then* $|B| \geq 3(q+1)/2$. *This bound is sharp.*
3. **(Blokhuis [5], Blokhuis et al. [6])** *If* $q=p^{2e+1}$, *p prime,* $e \geq 1$, *then* $|B| \geq \max(q+1+p^{e+1}, q+1+c_p q^{2/3})$, *where c_p equals* $2^{-1/3}$ *if* $p \in \{2,3\}$ *and* 1 *if* $p \geq 5$.

However, it is not hard to see that if $q = 2$, then every blocking set in $\mathrm{PG}(2,q)$ is trivial. Hence the situation for nontrivial blocking sets with respect to t-spaces in $\mathrm{PG}(n,2)$ must be different from the situation described in Theorem 3.2. This case has been studied during the stay and some progress has been made in proving the following theorem. Since the proof is not yet finished, it will be stated as a conjecture.

Conjecture 3.4

1. In $\mathrm{PG}(n,2)$, $n \geq 3$, the smallest nontrivial blocking sets with respect to hyperplanes are *skeletons* of a solid in $\mathrm{PG}(n,2)$; these are sets of five points in a 3-space no four of which are coplanar. If $n = 3$, then these are the only minimal nontrivial blocking sets with respect to planes.

2. Up to isomorphism, there is only one nontrivial minimal blocking set with respect to lines in $\mathrm{PG}(3,2)$. It consists of ten points and can be described as $l \cup l_1 \cup l_2 \cup l_3$, where l_1, l_2 and l_3 are three concurrent, not coplanar lines skew to the line l.

3. In $\mathrm{PG}(n,2)$, $n \geq 3$, the smallest nontrivial blocking sets with respect to t-spaces, $1 \leq t \leq n-2$, have size $2^{n-t+1} + 2^{n-t-1} + 2^{n-t-2} - 1$ and are cones with vertex an $(n-t-3)$-space π_{n-t-3} and base a nontrivial minimal blocking set with respect to lines in a solid skew to π_{n-t-3}.

Remark 3.5

- In [3], Beutelspacher notes that a skeleton of a 3-space in $\mathrm{PG}(n,2)$ is a nontrivial blocking set with respect to hyperplanes.
- Parts 1 and 2 of Conjecture 3.4 have already been proved and some lemmas have been proved to facilitate proving part 3 of the conjecture.

3.2 Partial hemisystems

Let $\mathrm{Q}(4,q)$ denote the *nonsingular quadric* in $\mathrm{PG}(4,q)$ and $\mathrm{W}_3(q)$ the threedimensional *symplectic space* over $\mathrm{GF}(q)$.

A *hemisystem* $\mathcal{H}$ on $\mathrm{Q}(4,q)$ is a set of points on $\mathrm{Q}(4,q)$ such that each line of $\mathrm{Q}(4,q)$ contains exactly $(q+1)/2$ points of $\mathcal{H}$. If $\mathcal{H}$ is a hemisystem of $\mathrm{Q}(4,q)$, then $|\mathcal{H}| = (q+1)(q^2+1)/2$. A *partial hemisystem* $\mathcal{H}$ on $\mathrm{Q}(4,q)$ is a set of points on $\mathrm{Q}(4,q)$ such that each line of $\mathrm{Q}(4,q)$ contains at most $(q+1)/2$ points of $\mathcal{H}$. The *deficiency* δ of a partial hemisystem $\mathcal{H}$ of $\mathrm{Q}(4,q)$ is by definition the number of points it lacks to be a hemisystem, whence $\delta = (q+1)(q^2+1)/2 - |\mathcal{H}|$.

Since $\mathrm{Q}(4,q)$ is the point-line dual of $\mathrm{W}_3(q)$, see e.g. [17, §3.2], it makes sense to introduce the dual notion: a *(partial) dual hemisystem* $\mathcal{H}^*$ on $\mathrm{W}_3(q)$ is a set of lines on $\mathrm{W}_3(q)$ such that each point of $\mathrm{W}_3(q)$ is incident with (at most) $(q+1)/2$ lines of $\mathcal{H}^*$. The *deficiency* δ of a partial dual hemisystem equals $(q+1)(q^2+1)/2 - |\mathcal{H}^*|$.

In the beginning of Section 3, the definition of a minihyper was given. However, the definition given there was the one for a minihyper without weights. But in the context of dual hemisystems, we will need the concept of a minihyper in its full generality.

An $\{f,m;n,q\}$-*minihyper* is a pair (F,w), where F is a subset of the point set of $\mathrm{PG}(n,q)$ and w is a weight function $w:\mathrm{PG}(n,q)\to\mathbb{N}:P\mapsto w(P)$, satisfying (i) $w(P)>0\Leftrightarrow P\in F$, (ii) $\sum_{P\in F}w(P)=f$, and (iii) $\min\{\sum_{P\in H}w(P): H \text{ is a hyperplane}\}=m$.

The definition from the beginning of Section 3 is a special case of this one. Namely, if the weight function w from the general definition is a mapping onto $\{0,1\}$, the minihyper (F,w) can be identified with its set F of points with weight one and the original definition of a minihyper is obtained as a special case of the more general definition presented here.

The relation between minihypers and dual hemisystems is presented in the following two theorems, which were proved in collaboration with Prof. Dr. Leo Storme (Ghent University) during the visit. In these theorems, the following notation is used: if P is a point of the projective space, then $\mathsf{star}(P)$ denotes the set of all lines through P.

Theorem 3.6 (with L. Storme) *Suppose $\mathcal{H}^*$ is a partial dual hemisystem of $\mathrm{W}_3(q)$ with deficiency δ. Define a weight function w as follows:*

$$w:\mathrm{PG}(3,q)\to\mathbb{N}:P\mapsto\frac{q+1}{2}-|\mathsf{star}(P)\cap\mathcal{H}^*|.$$

If F is the set of points of $\mathrm{PG}(3,q)$ with positive weight, then (F,w) is a $\{\delta(q+1),\delta;3,q\}$-minihyper.

The minihypers that occur in the statements of Theorem 3.6 belong to that special class of minihypers that are equivalent to linear codes meeting the Griesmer bound, see Section 2. Minihypers of this type have enjoyed special attention by researchers for the obvious reason that they are related to codes. In [13], a result is proved that perfectly fits this case. This theorem can be applied immediately to Theorem 3.6 to shed some light on the distribution of the points that are *not sufficiently covered by lines of $\mathcal{H}^*$*. For $q=2$, let ε_q equal 2. For $q>2$, let $q+\varepsilon_q$ denote the size of the smallest nontrivial blocking sets in $\mathrm{PG}(2,q)$; see Theorem 3.3 for some lower bounds on ε_q.

Corollary 3.7 *Suppose $\mathcal{H}^*$ is a partial dual hemisystem of $\mathrm{W}_3(q)$ with deficiency $\delta<\varepsilon_q$. If w is defined as in Theorem 3.6 and F is the set of points of $\mathrm{PG}(3,q)$ with positive weight, then (F,w) is a sum of δ lines.*

Remark 3.8 We have proved a similar theorem for partial dual hemisystems of the threedimensional *Hermitian variety* $\mathrm{H}(4,q^2)$ over $\mathrm{GF}(q^2)$. For definitions and the exact statement of the theorem, see the attached note *On partial hemisystems*.

3.3 Partial t-spreads

As mentioned above, a *partial t-spread* of $\mathrm{PG}(n,q)$ is a set of mutually disjoint t-spaces in $\mathrm{PG}(n,q)$. It is called *maximal* if no t-space can be added to obtain a larger partial t-spread.

During my visit, I was interested in the following question: "What is the size of the smallest maximal partial t-spreads of $\mathrm{PG}(n,q)$?" and have obtained upper and lower bounds on this size, see Theorem 3.9, which in a few cases are sharp and in many cases reasonably close to one another. Under some extra assumptions (which might never be fulfilled), Theorem 3.13

improves upon Theorem 3.9 and increases the number of cases in which the bounds are sharp significantly.

From the definition of maximality, it follows that a partial t-spread $\mathcal{S}$ in $\mathrm{PG}(n,q)$ is maximal if and only if the set of points covered by $\mathcal{S}$ is a blocking set with respect to t-spaces. Hence, in order to construct a small maximal partial t-spread, it makes sense to start from a small blocking set with respect to t-spaces and to try to find a partial t-spread which covers all its points and as little extra points as possible.

The smallest blocking sets with respect to t-spaces are characterised in Theorems 3.2 and 3.3 and in Conjecture 3.4. The smallest ones are $(n-t)$-spaces. Hence, a natural approach is to start from a maximal partial t-spread of such an $(n-t)$-space in $\mathrm{PG}(n,q)$ which is as large as possible and to find mutually disjoint t-spaces that cover the remaining points of this $(n-t)$-space. It is in this way that large maximal partial t-spreads become interesting for the determination of small maximal partial t-spreads. Upper bounds on the size of maximal partial t-spreads are known [2, 9], but these upper bounds are not always sharp. The largest known examples are constructed by Beutelspacher [2]. Using this construction in the $(n-t)$-spaces and applying the above reasoning yields reasonably small maximal partial t-spreads in $\mathrm{PG}(n,q)$.

Using the bounds on the size of nontrivial blocking sets with respect to t-spaces it is possible to prove that a maximal partial t-spread not covering an $(n-t)$-space always consists of more t-spaces than the example constructed. Using the upper bound on the size of maximal partial t-spreads, it is possible to obtain a lower bound for the size of maximal partial t-spreads whose elements cover an $(n-t)$-space.

Combining the observations above, the following result is obtained.

Theorem 3.9 *Let $s(t,n,q)$ denote the size of the smallest maximal partial t-spreads in $\mathrm{PG}(n,q)$ and write $n=k(t+1)+t-1+r$, $0\le r\le t$. Let $\beta=\lceil (t+r-1)/2\rceil$. If $k\ge 2$, then the following hold.*

1. *If $r=0$, then $s(t,n,q)=\frac{q^{k(t+1)}-1}{q^{t+1}-1}$.*
2. *If $r=1$, then $q\frac{q^{k(t+1)}-1}{q^{t+1}-1}+q^2-q+1\le s(t,n,q)\le q\frac{q^{k(t+1)}-1}{q^{t+1}-1}+q^{\beta+1}-q+1$.*
3. *If $r>1$ and $t+1\ge 2r$, then $q^r\frac{q^{k(t+1)}-1}{q^{t+1}-1}+(q^{r+1}-q^r)/2+1\le s(t,n,q)\le q^r\frac{q^{k(t+1)}-1}{q^{t+1}-1}+q^{\beta+1}-q^r+1$.*
4. *If $r>1$ and $t+1<2r$, then $q^r\frac{q^{k(t+1)}-1}{q^{t+1}-1}+(q^{r+1}-q^r+q^{2r-t-1}+3q+1)/2\le s(t,n,q)\le q^r\frac{q^{k(t+1)}-1}{q^{t+1}-1}+q^{\beta+1}-q^r+1$.*

If $\mathcal{S}$ is a maximal partial t-spread in $\mathrm{PG}(n,q)$ whose size lies in the corresponding interval above, then $\cup_{\pi_t\in\mathcal{S}}\pi_t$ contains an $(n-t)$-space in $\mathrm{PG}(n,q)$.

Corollary 3.10 1. *In $\mathrm{PG}(2k+1,q)$, $k\ge 2$, the smallest maximal partial linespreads have size $q\frac{q^{2k}-1}{q^2-1}+q^2-q+1$.*

2. *In* $\mathrm{PG}(3k+2,q)$*,* $k \geq 2$*, the smallest maximal partial planespreads have size* $q\frac{q^{3k}-1}{q^3-1} + q^2 - q + 1$.

Corollary 3.11 *If* $n \neq 3$*, then the size of the smallest maximal partial linespreads in* $\mathrm{PG}(n,q)$ *is known.*

As mentioned above, the upper bounds on the size of maximal partial t-spreads in [2, 9] are not sharp. However, there is a conjecture regarding the exact upper bound (it states that it is the size of largest known examples).

Conjecture 3.12 (Eisfeld and Storme [10]) If $n+1 = k(t+1)+r$, $1 \leq r \leq t$, then the largest maximal partial t-spreads in $\mathrm{PG}(n,q)$ have size $q^r\frac{q^{k(t+1)}-1}{q^{t+1}-1} - q^r + 1$.

Assuming the correctness of this conjecture, the lower bounds from Theorem 3.9 can be improved upon and the following results can be obtained.

Theorem 3.13 *Let* $s(t,n,q)$ *denote the size of the smallest maximal partial* t*-spreads in* $\mathrm{PG}(n,q)$ *and write* $n = k(t+1)+t-1+r$*,* $0 \leq r \leq t$*. Let* $\beta = \lceil (t+r-1)/2 \rceil$ *and assume that Conjecture* 3.12 *is true. If* $k \geq 2$*, then the following hold.*

1. *If* $r = 0$*, then* $s(t,n,q) = \frac{q^{k(t+1)}-1}{q^{t+1}-1}$.

2. *If* $r > 0$*, then* $q^r\frac{q^{k(t+1)}-1}{q^{t+1}-1} + q^{r+1} - q^r + 1 \leq s(t,n,q) \leq q^r\frac{q^{k(t+1)}-1}{q^{t+1}-1} + q^{\beta+1} - q^r + 1$.

If $\mathcal{S}$ *is a maximal partial* t*-spread in* $\mathrm{PG}(n,q)$ *whose size lies in the corresponding interval above, then* $\cup_{\pi_t \in \mathcal{S}} \pi_t$ *contains an* $(n-t)$*-space in* $\mathrm{PG}(n,q)$.

Corollary 3.14 *Assume that Conjecture* 3.12 *is correct. Then the following hold.*

1. *In* $\mathrm{PG}((k+1)(t+1)+t-2,q)$*,* $k \geq 2$*, the smallest maximal partial* t*-spreads have size* $q^t\frac{q^{k(t+1)}-1}{q^{t+1}-1} + q^{t+1} - q^t + 1$.

2. *In* $\mathrm{PG}((k+1)(t+1)+t-3,q)$*,* $t > 1$*,* $k \geq 2$*, the smallest maximal partial* t*-spreads have size* $q^{t-1}\frac{q^{k(t+1)}-1}{q^{t+1}-1} + q^t - q^{t-1} + 1$.

Corollary 3.15 *If Conjecture* 3.12 *is true, then for* $n \notin \{5,6\}$ *the size of the smallest maximal partial planespreads in* $\mathrm{PG}(n,q)$ *is known.*

Remark 3.16 The results in this subsection are not yet entirely proved for $q = 2$. In this case they still depend on the outcome of the investigations from Subsection 3.1. If the conjecture stated there is correct then the results from this subsection also hold for $q = 2$. In fact, only a weaker version of the conjecture is needed: a nontrivial blocking set with respect to t-spaces, $1 \leq t < n-1$, in $\mathrm{PG}(n,2)$ has size at least $2^{n-t+1} + 2^{n-t-1} + 2^{n-t-2} - 1$.

3.4 Constructing sets of MAXMOLS

The results in this subsection are joint work with Prof. Dr. András Gács, Prof. Dr. Peter Sziklai and Prof. Dr. Tamás Szőnyi (Eötös Loránd University).

An $(s,r;1)$-*net*, $r \geq 3$, is an incidence structure $\mathsf{D} = (\mathsf{P}, \mathsf{B}, \mathtt{I})^4$ satisfying

1. the relation $\sim$ on B, with $l \sim m$ if $l = m$ or if there exists no point P such that $l \; \mathtt{I} \; P \; \mathtt{I} \; m$, is an equivalence relation which has r equivalence classes; these classes are called *parallel classes*;
2. for all $l, m \in \mathsf{B}$, if $l \not\sim m$, then there exists a unique point P such that $l \; \mathtt{I} \; P \; \mathtt{I} \; m$;
3. every point lies in an element of each parallel class.

If these properties hold, then there exists an integer s for which the following properties hold:

- the number of lines in a parallel class equals s;
- the number of lines through a point equals r;
- each line contains s points;
- there are s^2 points;
- there are rs lines.

If $s > 1$, then $r \leq s+1$. Equality holds if and only if D is an affine plane.

As mentioned before, an (s,r)-net immediately yields a set of $r-2$ MOLS(s) and vice versa. Here, (s,r)-nets will be constructed to obtain sets of MOLS, which after careful examination turn out to be sets of MAXMOLS.

There are different ways to construct (s,r)-nets. The approach followed here is to take as points the points of AG$(2,q)$ and as lines certain sets of q points. The parallel classes will be sets consisting of q translates of a line.

For the moment, our attention is restricted to the case where q is a prime. We construct our net as follows. The points of the net are the points of AG$(2,q)$. One parallel class of lines is defined by the vertical translates of the graph of the function $f : x \mapsto x^{(q+1)/2}$. The graph of this function determines $(q+3)/2$ directions. Denote this set of directions determined by f by D_f. Now for each of the remaining directions $d \in \mathrm{GF}(q)^+ \setminus D_f$, where $\mathrm{GF}(q)^+ = \mathrm{GF}(q) \cup \{\infty\}$, add the set of lines with direction d as an extra parallel class. In this way a $(q,(q+1)/2)$-net is obtained and hence a set of $(q-3)/2$ MOLS(q). As mentioned in Section 2 proving the maximality of a net is often the difficult part and this case is no exception. Fortunately, if q is a prime, theorems of Rédei [18] and Lovász and Schrijver [16] on functions determining specific sets of directions help a lot to ease this task and the maximality of the net can be proved.

Theorem 3.17 (with A. Gács, P. Sziklai and T. Szőnyi) *For $p > 2$ prime, there exist $(p-3)/2$* MAXMOLS(p).

Remark 3.18 For $p \geq 13$, these MAXMOLS are new.

[4] The elements of P are called *points*, the elements of B are called *lines* and $\mathtt{I} \subseteq (\mathsf{P} \times \mathsf{B}) \cup (\mathsf{B} \times \mathsf{P})$ is a symmetric incidence relation.

4 Other activities

During my stay in Budapest I had the pleasure of having several interesting discussions which might lead to future research projects:

- with Prof. Dr. Tamás Szőny (SZTAKI and ELTE) and Dr. Gerzson Kéri (SZTAKI) on a paper that Dr. Kéri is writing on the classification of small MDS-codes. As from this classification he is able to draw conclusions on small arcs in projective spaces, it is interesting to compare the results he obtains with the literature on arcs and see in which cases these results are new and/or give some clues as to the situation for general n and q.
- with Prof. Dr. Tamás Szőny on functions in $\mathrm{AG}(2,q)$ determining a small number of directions and on constructions for sets of MAXMOLS using spreads and ovoids on finite classical polar spaces.
- with Prof. Dr. András Gács (ELTE) and Prof. Dr. Peter Sziklai (ELTE) on the topic of Cameron-Liebler line classes in $\mathrm{PG}(3,q)$.
- with Prof. Dr. András Gács on the construction of maximal partial spreads of different sizes in $\mathrm{PG}(3,q)$.

I also had the opportunity to attend several interesting lectures at SZTAKI, at the Alfréd Rényi Institute of Mathematics of the Hungarian Academy of Sciences and at the Eötvös Loránd University.

During the stay, I gave three talks:

- Mixed partial spreads and MAXMOLS, Seminar on Finite Geometry at ELTE, 14 November 2003.
- Codes, spreads and Latin squares, SZTAKI, 21 November 2003.
- Applications of Ball's 1 mod p result for ovoids of $\mathrm{Q}(4,q)$, Seminar on Finite Geometry at ELTE, 28 November 2003.

5 Attachments

The following items are attached to this report.

- A preliminary version for a paper on *Small maximal partial t-spreads in* $\mathrm{PG}(n,q)$, the research for which was done during my stay in Budapest.
- Some notes on *Nontrivial blocking sets in* $\mathrm{PG}(n,2)$, research that was initiated during the stay and that will be continued.
- Some notes on *Partial hemisystems*, joint work with Prof. Dr. Leo Storme, part of which was done during the stay.

- A printout of the slides for *Codes, spreads and Latin squares*, a talk presented at the SZTAKI Centre of Excellence in Information Technology and Automation on 21 November 2003. The original slides are in colour and can be found at the URL

  `http://www.sztaki.hu/infolab/ferret-govaerts03.html`

  of the web page announcing the talk.

- Lecture notes for *Mixed partial spreads and MAXMOLS*, a talk presented at the Seminar on Finite Geometry at Eötvös Loránd University on 14 November 2003.

- Lecture notes for *Applications of Ball's* 1 *mod p result for ovoids of* $Q(4,q)$, a talk presented at the Seminar on Finite Geometry at Eötvös Loránd University on 28 November 2003.

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I am grateful to the Computer and Automation Research Institute of the Hungarian Academy of Sciences for presenting me the opportunity to work at the institute, for the excellent research environment and for the hospitality and financial support extended to me. It was a pleasure to work in the beautiful city of Budapest. I will certainly recommend and encourage other researchers to grab the opportunity if it is presented to them.

Patrick Govaerts
Budapest, 29 January 2004

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