1 Data of the visit

Visitor

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PhD

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Title: Projective Spaces and Linear Codes
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Area: Finite geometry, Coding theory
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2 Research topic

In many situations, it is necessary to transmit data from one place to an other. A classical example is the transmission of data via satellites. Also playing music on a compact disc is an example of data transmission; the music stored in a binary form on the compact disc is transmitted via a laser beam to the compact disc player and reconverted into music.

During the transmission of data, it is possible that there occur *errors.* Atmospheric circumstances can change data in satellite communication; scratches and finger prints on a compact disc can make the binary information stored on the disc unreadable.

The crucial question is how these inevitable errors can be *corrected.*

The solution is to construct *codes.* For practical purposes, we can restrict ourselves to *linear codes.*

Linear codes are described by their four parameters: we speak of $[n, k, d; q]$-codes. These are k-dimensional subspaces of the n-dimensional vector space $V(n, q)$ over the finite field $GF(q)$, such that two distinct vectors of these subspaces differ in at least d positions. For fixed value of q, the order of the underlying field, the best linear codes are codes with:
(1) short *length* n, for fast data transmission,
(2) large *dimension* k, for a large variety of information that can be transmitted,
(3) large *minimal distance* d, for being able to correct a lot of errors.

Research in *coding theory* starts with constructing *good* codes. When a parameter n, k or d is *optimalized* for fixed values of the field order q and the other two parameters, we say that the code is *optimal*.

Within this project, coding-theoretical problems will be studied by looking at the equivalent problems regarding substructures of finite projective spaces, so of substructures in Galois geometries. Namely, the *linear codes* are the classical examples of practical applications of *Galois geometries*. Galois geometries consist of the study of the properties of finite projective spaces. Many problems in Galois geometries have a meaning as problem in coding theory, and vice versa.

This means that a lot of problems in coding theory can be studied from a geometrical point of view, giving, next to the techniques of coding theory, geometrical methods to study these problems. These geometrical methods have led to important results in coding theory.

Project

This project has as a global objective to study the link between the following two disciplines:

(1) the theory of linear codes, and

(2) Galois geometries.

For a linear $[n, k, d; q]$-code, we have the following bound [3], known as the *Griesmer bound*: $n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil = g_q(k, d)$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

We want to study the relationship between *minihypers* (which are certain subsets of Projective Spaces; see further) and the theory of *linear codes meeting the Griesmer bound*. This study has already been started in previous articles. We address several specific questions which then occurred and which will allow us to obtain a deeper insight in the relationship between Galois geometries and the theory of linear codes. An example of such a question is the (non)-existence of linear binary codes attaining the Griesmer bound. A partial answer, namely for *projective* binary codes (these are codes for which no two columns of the generator matrix are a multiple of each other, hence codes having dual distance at least 3), was given by T. Helleseth [4]. Following the advice of T. Helleseth, a systematic study of *non-projective* linear codes meeting the Griesmer bound was initiated. To continue this research, new results on minimal blocking sets are needed.

The results which will be obtained also have a lot of applications in other research areas: for instance, on extendability of *partial spreads* and *ovoids* in *projective spaces*, in *polar spaces* and in the *generalized hexagon* $H(q)$. We state these results with the bounds and conditions of [1]

Application 2.1 *We classified all $[n, k, d; q]$-codes with*

$$d = \theta q^{k-1} - \delta q^{\mu}, \qquad n = \theta v_k - \delta v_{\mu+1}; \tag{1}$$

for $\delta < 2p^2 - 4p$ (and all $\mu \leq k-1$), and total excess $\leq p^2 + p$. (θ is the maximum weight of a point).

Application 2.2 *An* s-spread *of $PG(N, q)$ is a set of s-spaces which partition $PG(N, q)$.*

A partial s-spread *of $PG(N, q)$ is a set of mutually skew s-spaces in $PG(N, q)$.*

A partial spread S *is* maximal *if it cannot be extended to a larger (partial) spread.*

The holes of S *are the points of $PG(N, q)$ not in an element of S.*

If $|S| = v_{N+1}/v_{s+1} - \delta$, then we call δ the deficiency *of S.*

Theorem *(Govaerts and Storme)*

If $(s+1)|(N+1)$ and $\delta < q$, the holes of S form a $\{\delta v_{s+1}, \delta v_s; N, q\}$-minihyper

F.

Theorem
Let S be a maximal partial μ-spread in $PG(t, p^3)$, $(\mu+1)|(t+1)$, with deficiency $\delta \leq 2p^2 - 4p$; $p = p_0^h$, p_0 prime, $p_0 \geq 7$, $p \geq 9$, $h \geq 1$.

Then $\delta = r(p^{3/2}+1) + s(p^2+p+1)$, where $r \in \mathbb{N}$ and $s \in \{0, 1\}$.

Moreover, the case $(r, s) = (1, 0)$ cannot occur when $(2\mu+1)(p^{3/2}-1) < q-1$ [Blokhuis-Metsch].

Application 2.3 *Let $\mathcal{P}$ denote a finite classical polar space.*
A μ-spread *of $\mathcal{P}$ is a set of totally isotropic μ-dimensional subspaces of $\mathcal{P}$ partitioning the point set of $\mathcal{P}$.*
A partial μ-spread *of $\mathcal{P}$ is a set of pairwise disjoint totally isotropic μ-dimensional subspaces.*

Necessary conditions [Govaerts-Storme] for the existence of a μ-spread are:
(1) for $W_{2n+1}(q)$: $(\mu+1)|(n+1)$,
(2) for $Q(2n, q)$: $(\mu+1)|(2n)$,
(3) for $Q^+(2n+1, q)$: $(\mu+1)|(n+1)$,
(4) for $Q^-(2n+1, q)$: $(\mu+1)|n$,
(5) for $H(2n, q^2)$: $(\mu+1)|n$ and,
(6) for $H(2n+1, q^2)$: $(\mu+1)|(n+1)$.

If these conditions are satisfied, we say that the size of $\mathcal{P}$ admits a μ-spread.

The deficiency *of a partial μ-spread S of $\mathcal{P}$ is the size of a hypothetical μ-spread of $\mathcal{P}$ minus $|S|$. A* hole *of a partial μ-spread S of $\mathcal{P}$ is a point of $\mathcal{P}$ not contained in an element of S.*

Theorem *(Govaerts and Storme)*
If $\mathcal{P}$ is a classical polar space in $PG(t, q)$ whose size admits a μ-spread and if S is a partial μ-spread of $\mathcal{P}$ with deficiency $\delta < q$, then the set of holes of S forms a $\{\delta v_{\mu+1}, \delta v_\mu; t, q\}$-minihyper.

A generator *is a maximal totally isotropic subspace of $\mathcal{P}$.*

Lemma
(1) *A (projected) Baer subspace whose point set is contained in a quadric $\mathcal{P}$ is contained in a generator of $\mathcal{P}$.*
(2) *A (projected) subspace over $GF(\sqrt[3]{q})$ contained in a quadric $\mathcal{P}$ is contained in a generator of $\mathcal{P}$.*

Theorem
Let $q = p^3$, $p = p_0^h$, p_0 *prime,* $p_0 \geq 7$, $p \geq 9$, $h \geq 1$. *Assume* $\mathcal{P}$ *is a nonsingular quadric whose size admits a* μ*-spread, and with* $2\mu + 1$ *larger than the dimension of a generator of* $\mathcal{P}$.

If $\delta \leq 2p^2 - 4p$ *is the deficiency of a partial* μ*-spread* S *of* $\mathcal{P}$*, then* S *is extendable to a* μ*-spread of* $\mathcal{P}$.

Application 2.4 *Let* $x(x_0, \ldots, x_6)$ *and* $y(y_0, \ldots, y_6)$ *be two different points in* $PG(6, q)$. *The* Grassmann coordinates *of the line* xy *are* $(p_{01}, p_{02}, \ldots, p_{56})$*, where*

$$p_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$$

The split Cayley hexagon *is a point-line incidence structure* $H(q)$ *defined in the following way. The* points *of* $H(q)$ *are the points of* $PG(6, q)$ *lying on the quadric* $Q(6, q)$ *with equation* $X_0X_4 + X_1X_5 + X_2X_6 = 0$. *The* lines *of* $H(q)$ *are the lines of this quadric whose Grassmann coordinates satisfy* $p_{12} = p_{34}$, $p_{54} = p_{32}$, $p_{20} = p_{35}$, $p_{65} = p_{30}$, $p_{01} = p_{36}$, $p_{46} = p_{31}$. Incidence *is containment.*

Two points of $H(q)$ *are called* opposite *if they are at distance 6 from each other in the incidence graph of* $H(q)$ *(this is the maximal possible distance).*

An ovoid *of* $H(q)$ *is a set of* $q^3 + 1$ *mutually opposite points. If* $\mathcal{O}$ *is an ovoid of* $H(q)$*, then the set of* $q^3 + 1$ *planes* $x^\perp$*, with* $x \in \mathcal{O}$*, forms a plane spread of* $Q(6, q)$ *[Van Maldeghem]. A* partial ovoid *of* $H(q)$ *is a set of mutually opposite points of* $H(q)$*, and it is called* maximal *if it cannot be extended to a larger set of mutually opposite points. The* deficiency *of a partial ovoid of* $H(q)$ *containing* N *points is* $\delta = q^3 + 1 - N$.

As a corollary from the classification of minihypers, we have

Corollary
Assume $q = p^3$, $p = p_0^h$, $p_0 \geq 7$ *prime,* $h \geq 1$, $p \geq 9$. *If* $\mathcal{O}$ *is a partial ovoid of* $H(p^3)$ *of deficiency* $\delta \leq 2p^2 - 4p$*, then* δ *is even.*

Research performed

Definition 2.1 *A* blocking set *of $PG(2, q)$ is a set of points intersecting every line of $PG(2, q)$ in at least one point.*

A blocking set is called minimal *when no proper subset of it is still a blocking set; we call a blocking set* trivial *when it contains a line; and a blocking set is called* linear *if it is a subgeometry or a projected subgeometry.*

The study of linear blocking sets will have a geometric approach, the main objective of this study is to obtain a good understanding of the geometric structure of these objects. This work is a starting point for the construction and characterization of the minihypers; and hence of linear codes meeting the Griesmer bound.

The aim of this study is to obtain: (1) combinatorial results on minimal blocking sets (in the sense of [2]), and (2) structural characterizations of small minimal blocking sets (as in [5]).

When the results described above are obtained, they will be applied to construct and classify *minihypers*.

Definition 2.2 *An* $\{f, m; N, q\}$-minihyper *is a pair (F, w), where F is a subset of the point set of $PG(N, q)$ and w is a weight function $w : PG(N, q) \to \mathbb{N} : x \mapsto w(x)$, satisfying*
(1) $w(x) > 0 \Leftrightarrow x \in F$,
(2) $\sum_{x \in F} w(x) = f$, *and*
(3) $\min(|(F, w) \cap H| = \sum_{x \in H} w(x) \mid H \in \mathcal{H}) = m$; *where $\mathcal{H}$ denotes the set of hyperplanes of $PG(N, q)$.*

The classifications of these minihypers are equivalent to classifications of linear codes meeting the Griesmer bound; if the minihyper does not exist, we know that for the considered parameters, there is no linear code meeting the Griesmer bound.

The classifications of the minihypers are by induction on the dimension; therefore it is particularly recommended to study the blocking sets, since they appear as planar intersections with the minihypers.

We now describe how we will also included *non-projective* linear codes in the classifications.

A further reason to study minihypers with weights, or equivalently non-projective linear codes meeting the Griesmer bound, is that we can describe projected subgeometries appearing in such minihypers in a very natural way. Namely in the following way:

Let Π be a projected $PG(d, p^t)$. The weight *of a point s of Π is the number of points s' of $PG(d, p^t)$ that are projected onto s.*

We refer to the attached article [1] for a detailed description of the research performed.

3 Work in progress

A starting point is the article [1], which was finnished during the research visit. The next step is to use these results as an induction basis, for both the dimension of the projective space, as for the size of the minihypers. Hence, we want a characterization of $\{\delta v_{\mu+1}, \delta v_\mu; N, p^3\}$-minihypers.

We now describe the steps which will be needed to obtain our final characterization.

Let F be a $\{\delta(p^3+1), \delta; N, p^3\}$-minihyper.

We suppose the total excess $\sum_{x\in F}(w(x)-1)$ is at most p^3-4p and $\delta \leq 2p^2-4p$.

Step 1

A $\{\delta(p^3+1), \delta; N, p^3\}$-minihyper F, $N \geq 4$, p non-square, $p = p_0^h$, p_0 prime, $h \geq 1$, $p_0 \geq 7$, $\delta \leq 2p^2-4p$, and with excess $e \leq p^3-4p$, is either:
(1) a sum of lines and at most one (projected) $PG(5, p)$,
(2) a sum of lines and a $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper $\Omega \setminus N$, where Ω is a $PG(5, p)$ projected from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 3$, and where N is the line contained in Ω.

Step 2

Let F be a $\{\delta(p^6+p^3+1), \delta(p^3+1); N, p^3\}$-minihyper, $\delta \leq 2p^2-4p$, $N \geq 3$, $p \geq 9$ non-square, $p = p_0^h$, $h \geq 1$, $p_0 \geq 7$ prime, with excess $e \leq p^2+p$.

Then F is a sum of planes and at most one projected subgeometry $PG(8, p)$.

Step 3

Let F be a $\{\delta v_{\mu+1}, \delta v_\mu; N, p^3\}$-minihyper, $\mu \geq 3$, $\delta \leq 2p^2-4p$, $N \geq 3$, $p = p_0^h$, $h \geq 2$ even, $p_0 \geq 7$ prime, with excess $e \leq p^2+p$.

Then F is a sum of μ-dimensional spaces $PG(\mu, p^3)$, (projected) $PG(2\mu+1, \sqrt{q})$'s, and of at most one (projected) subgeometry $PG(3\mu+2, p)$.

4 Future collaborations

4.1 On general blocking sets

A *t-fold* $(n-k)$*-blocking set* in $PG(n, q)$ is a set B of points of $PG(n, q)$ intersecting every k-dimensional subspace in at least t points. A t-fold $(n-k)$-blocking set B of $PG(n, q)$ is called *minimal* when no proper subset of B is still a t-fold $(n-k)$-blocking set.

A 1-fold $(n-k)$-blocking set of $PG(n, q)$ is also simply called an $(n-k)$-*blocking set* of $PG(n, q)$.

We will obtain new classification results on minimal t-fold $(n-k)$-blocking sets in $PG(n,q)$.

We will first extend the following $1 \pmod p$ result of Szőnyi and Weiner [6] to a $t \pmod p$ result on minimal t-fold $(n-k)$-blocking sets in $PG(n,q)$.

Theorem 4.1 *Let B be a minimal $(n-k)$-blocking set in $PG(n,q)$, $q=p^h$, $p>2$ prime, $h\geq 1$, of size less than $3(q^{n-k}+1)/2$. Then every subspace that intersects B in at least one point, intersects B in $1 \pmod p$ points.*

Theorem 4.2 *Let B be a minimal t-fold $(n-k)$-blocking set in $PG(n,q)$, $q=p^h$, $p>2$ prime, $h\geq 1$, of size less than $(t+3/2)(q^{n-k}+1)$. Then every k-dimensional subspace intersects B in $t \pmod p$ points, and any subspace of dimension less than k intersects B in $0,1,\ldots,t \pmod p$ points.*

Such $1 \pmod p$ results, or more general $t \pmod p$ results, also appear with respect to other substructures in finite projective spaces, and are very useful for obtaining classification theorems on these substructures.

We present how this $t \pmod p$ result implies new classification results on minimal t-fold $(n-k)$-blocking sets in $PG(n,q)$. The most general results are obtained when q is square. We will give classification results on minimal t-fold $(n-k)$-blocking sets in $PG(n,q)$, q square, containing subspaces $PG(k,q)$, but also, possibly projected, subgeometries over a subfield.

4.2 Blocking sets applied to minihypers

The further step in characterising the minihypers will be done by making use of the previously mentioned results.

Future characterizations

Minihypers are (non)-minimal multiple $(N-r)$-blocking sets.

Example 4.3 *A disjoint plane and a line in $PG(4,q)$ are*
a non-minimal 2-blocking set,
a minimal $(q+2)$-fold 1-blocking set =
a $\{v_3+v_2, v_2+v_1; 4, q\}$-minihyper.

We give some examples of preliminary theorems which make use of the results on blocking sets. These theorems are joint work of Sziklai and Weiner (Budapest university), the visitor and Storme.

Lemma 4.4 *A $\{t(q^2+q+1)+\epsilon_1(q+1)+\epsilon_0, t(q+1)+\epsilon_1; 4, q\}$-minihyper with $t>1$, $t\leq p/2$, $\epsilon_1+\epsilon_0\leq t\sqrt{q}$ contains a minimal t-fold 2-blocking set.*

Theorem 4.5 (Ferret, Storme, Sziklai and Weiner) *If $t > 1$, $t \leq p/2$, then a t-fold 2-blockig set in $PG(4,q)$ is the disjoint union of $PG(4,\sqrt{q})$ and/or Baer cones.*

Theorem 4.6 (Ferret, Storme, Sziklai and Weiner) *A $\{t(q^2+q+1)+\epsilon_1(q+1)+\epsilon_0, t(q+1)+\epsilon_1; 4, q\}$-minihyper with $t > 1$, $t \leq p/2$, $\epsilon_1+\epsilon_0 \leq t\sqrt{q}$ is the disjoint union of t $PG(4,\sqrt{q})$.*

5 Attachments

1. Article obtained during the research visit.
2. Slides of the talk given at Sztaki on November 21, 2003.
3. Slides of the talk given at Eötvös Loránd University on November 21, 2003.

References

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[3] J.H. Griesmer, A bound for error-correcting codes. *IBM J. Res. Develop.* **4** (1960), 532-542.

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[5] O. Polverino and L. Storme, Minimal blocking sets in $PG(2,q^3)$. *Europ. J. Combin.* **23**(1) (2002), 83–92.

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