ON THE EFFECT OF BOUNDARY CONDITIONS ON CNN DYNAMICS: STABILITY AND INSTABILITY, BIFURCATION PROCESSES AND CHAOTIC PHENOMENA

I. Petras[†], *P. Checco*[‡], *M. Gilli*[‡], *T. Roska*[†], *and M. Biey*[‡]

[†] Hungarian Academy of Sciences
Computer and Automation Research Institute
Budapest, Hungary

[‡] Politecnico di Torino, Dipartimento di Elettronica
Torino, Italy

gilli@polito.it *roska@sztaki.hu*

ABSTRACT

The effect of the boundary conditions on the global dynamics of cellular neural networks (CNNs) is investigated. As a case study one dimensional template CNNs are considered. It is shown that if off-diagonal template elements have the same sign (i.e. the CNN is completely stable), then the qualitative dynamics is not influenced by the boundary conditions. If the off-diagonal template elements have opposite sign, then the boundary conditions behave as bifurcation parameters and can give rise to a very complex dynamic behavior. In particular they determine the equilibrium point patterns, the transition from stability to instability and the occurrence of several bifurcation phenomena leading to strange and/or chaotic attractor.

1. INTRODUCTION

Cellular neural networks (CNNs) are analog dynamic processor arrays [1]-[3]. A CNN can be described as a 2 or 3-dimensional array of identical nonlinear dynamical systems (called cells), that are locally interconnected. This property has allowed the realization of several high-speed VLSI chips [4], [5]. In most applications the connections are specified through space-invariant templates (that consist of small sets of parameters identical for all the cells.)

The mathematical model of a CNN consists in a large set of coupled nonlinear differential equations, that have been mainly studied through extensive computer simulations. For what concerns the dynamic behavior, CNNs can be divided in two classes: stable CNNs, with the property that each trajectory (with the exception of a set of measure zero) converges towards an equilibrium point; unstable CNNs, that exhibit at least one attractor, that is not a stable equilibrium point. The stability results are summarized in [6]: some examples of unstable CNNs presenting either periodic or nonperiodic (even chaotic) attractors are shown in [7]-[10].

With the exception of some general results, concerning CNN stability, and of manuscript [11], most studies concentrated on the effect of the template on CNN dynamics, without considering the influence of external inputs and boundary conditions. In [11] it was shown that for certain class of CNNs, the stability properties might depend on boundary conditions.

In this manuscript we investigate the effect of constant boundary conditions on CNN global dynamic behavior. As a case study we consider a CNN described by a one-dimensional template. By exploiting results presented in a previous paper [12], we prove that if the elements, symmetric with respect to the central one, have the same sign then the CNN is completely stable for any external constant inputs and/or boundary conditions. If the above condition is not verified, we show that boundary conditions behave as bifurcation parameters and can give rise to a rather complex dynamic behavior. In particular the boundary conditions determine the equilibrium point patterns, the transition from stability to instability and the occurrence of several bifurcation phenomena leading to strange and/or chaotic attractor. This extends the results presented in [11], where only the stability properties were considered, and open the possibility of exploiting constant boundary conditions and constant external inputs for designing new CNN functionalities.

2. ONE-DIMENSIONAL TEMPLATE CNN

We consider a CNN composed by N cells and described by the following one-dimensional templates:

$$\boldsymbol{A} = [A_{-2}\ A_{-1}\ A_0\ A_1\ A_2] \qquad (1)$$

This work was supported in part by *Ministero dell'Istruzione, dell'Università e della Ricerca*, under the National Research Plan PRIN 2001 and by *Consiglio Nazionale delle Ricerche (CNR)*, under contract no. CNRG000D51.

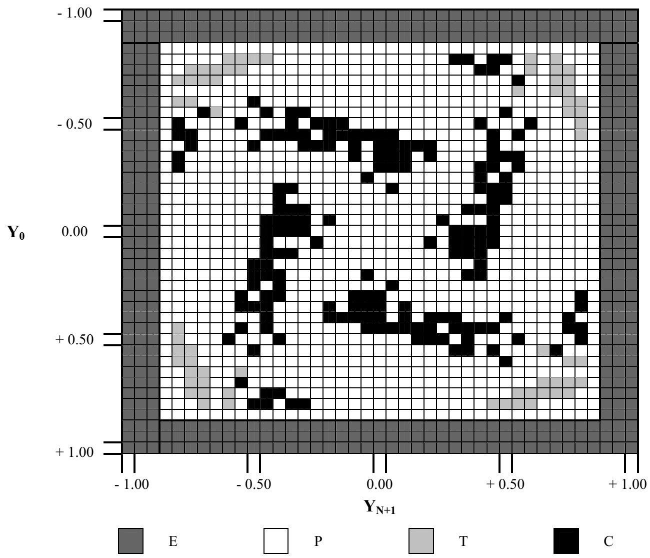

Figure 1: *Qualitative dynamic behavior of a CNN with* $N = 4$ *, described by the template* $s, p, -s$ *, with* $p = 1.1$ *and* $s = 0.9$*, as a function of the boundary conditions* Y_0 *and* Y_{N+1}*. The* E *subregions correspond to the existence of at least one stable equilibrium point. The* P *subregions correspond to the existence of only periodic attractors. The* T *subregions denote the existence of tori, whereas the* C *subregions show a chaotic behavior.*

$$\boldsymbol{B} = [B_{-2}\, B_{-1}\, B_0\, B_1\, B_2] \tag{2}$$

We denote by Y_{-1}, Y_0, and Y_{N+1}, Y_{N+2} the left and the right boundary conditions respectively, that are assumed to be constant.

The CNN dynamics is governed by the following normalized state equations:

$$\begin{aligned} \dot{x}_k \;=\; & -x_k \;+\; A_0\, y_k \;+\; A_{-2}\, y_{k-2} \;+\; A_{-1}\, y_{k-1} \\ & + A_1\, y_{k+1} \;+\; A_2\, y_{k+2} \\ & + B_0\, u_k \;+\; B_{-2}\, u_{k-2} \;+\; B_{-1}\, u_{k-1} \\ & + B_1\, u_{k+1} \;+\; B_2\, u_{k+2} \quad (1 \le k \le N) \end{aligned} \tag{3}$$

In the above equations x_k and u_k represent the state and the input voltage of the $k-th$ cell respectively (u_0, u_{-1}, u_1, and u_2 are assumed to be null). The output voltage y_k is defined as follows:

$$\begin{aligned} & y_{-1} = Y_{-1}\ ,\ y_0 = Y_0\ ,\ y_N = Y_N\ ,\ y_{N+1} = Y_{N+1} \\ & y_k = \tfrac{1}{2}\left(|x_k + 1| - |x_k - 1|\right) \quad (1 \le k \le N) \end{aligned} \tag{4}$$

3. ANALYSIS OF THE GLOBAL DYNAMIC BEHAVIOR

As a first step, we characterize the class of one-dimensional template CNNs, whose qualitative dynamics is not influenced by constant external inputs and boundary conditions. In order to do that we resort to the stability results presented in [12], that allow us to state the following Theorem.

Theorem 1: If the template elements of equations (3) satisfy the constraints $A_{-2}\ A_2 \ge 0$ and $A_{-1}\ A_1 \ge 0$, then the corresponding CNN is completely stable (i.e. all the trajectories converge towards an equilibrium point) for any external constant inputs and boundary conditions.

Proof : it is readily derived from Theorem 1 and Theorem 2 of [12].

If the conditions of the above Theorem are not satisfied, then external inputs and boundary conditions might alter the qualitative dynamics and play the role of bifurcation parameters.

For the sake of simplicity, we further restrict our attention to the following opposite-sign template $[s,\ p,\ -s]$ with $s > 0$ and $p - 1 < s$ and we assume that the external inputs be zero. The non-zero fixed boundary conditions are denoted by Y_0 and Y_{N+1}.

The results are summarized in Table 1 and Fig. 1. Table 1 reports the set of stable equilibrium point patterns that the CNN exhibits, as a function of the template parameters p and s and of the boundary conditions. It is seen that, if we exclude the case $|Y_0| < 1 - (p-1)/s$, $|Y_{N+1}| < 1 - (p-1)/s$, the network always exhibits at least one stable equilibrium point. In order to investigate the occurrence of a complex dynamic behavior, we concentrate on the parameter region corresponding to the absence of stable equilibrium points. Fig. 1 considers the template values $p = 1.1$ and $s = 0.9$ and the boundary conditions varying between -1 and $+1$ (i.e. $|Y_0| < 1$ and $|Y_{N+1}| < 1$), in a CNN with four cells. We have also verified that qualitative results similar to those shown in Fig. 1 are obtained using the same number of cells and different values of the template parameters p and s.

The parameter region (Y_0, Y_{N+1}) has been divided in several subregions and for each of them the CNN dynamic behavior has been simulated, through a numerical algorithm. The E subregions correspond to the existence of at least one stable equilibrium point $(1-(p-1)/s < |Y_0|,\ |Y_{N+1}| < 1)$; in such subregions we have also observed the coexistence of stable equilibrium points and stable limit cycles, that originate through some heteroclinic bifurcations. In the P subregions all the attractors are stable limit cycles; they normally originate through heteroclinic bifurcations. The T subregions correspond to the existence of tori, that bifurcate from stable limit cycles through Naimark-Sacker bifurcations. In the C subregions we have observed a chaotic behavior, that originates through the following routes: a sequence of period doubling bifurcations, for the C subregions close to the P ones; a sequence of Naimark-Sacker bifurcations (i.e. via torus breakdown) for the C subregions close to the T ones. The existence of chaotic attractors (that can also coexist) in the C subregions has also been verified through the computation of the Lyapunov exponents.

Summarizing, we have shown that for certain classes of templates (i.e. those not satisfying the conditions of Theorem 1) the introduction of constant boundary conditions may determine a complex and rich dynamic behavior, including several bifurcation processes. We remark that similar and even richer phenomena can be observed by introducing constant external inputs.

4. CONCLUSION

We have investigated the effect of constant boundary conditions of CNN global dynamic behavior. As a case study we have considered CNNs described by one-dimensional templates. By exploiting the results presented in [12], we have proved that if the off-diagonal elements have the same sign, the CNN is completely stable for any external constant inputs and/or boundary conditions. Then we have shown that in a CNN described by an opposite-sign template, the boundary conditions behave as bifurcation parameters and can give rise to a rather complex dynamic behavior. In particular, we have observed the occurrence of several bifurcation processes: equilibrium point local bifurcations (that determine the equilibrium point space-distribution); heteroclinic global bifurcations (that determine the transition from stability to instability); Neimark-Sacker, period doubling and torus breakdown bifurcations, leading to tori and/or chaotic attractor.

The results presented in the manuscript open the possibility of exploiting constant boundary conditions and constant external inputs for designing new CNN functionalities.

5. REFERENCES

[1] L. O. Chua and L. Yang, "Cellular Neural Networks: Theory," *IEEE Trans. Circuits and Syst.*, vol. 35, no. 10, pp. 1257-1272, Oct. 1988.

[2] L. O. Chua and T. Roska, "The CNN paradigm," *IEEE Trans. Circuits and Syst.-I*, vol. 40, no. 3, pp. 147-156, March 1993.

[3] L. O. Chua and T. Roska, *Cellular Neural Networks and Visual Computing*, Cambridge University Press, U.K., 2002.

[4] Á. Rodríguez-Vázquez, M. Delgado-Restituto, E. Roca, G. Linan, R. Carmona, S. Espejo and R. Dominguez-Castro,"CMOS Analogue Design Primitives," *Towards the visual microprocessor - VLSI design and use of Cellular Network Universal Machines*, pp. 87-131, Chichester, Ed. by T.Roska and A. Rodriguez-Vazquez, J.Wiley, 2000.

[5] M. Salerno, F. Sargeni, V. Bonaiuto, "A Dedicated Multi-Chip Programmable System for Cellular Neural Networks," *Analog Integrated Circuits and Signal Processing*, Kluwer Academic Publisher , Vol. 18, no. 2/3 , pp. 277-288, 1999.

[6] P. P. Civalleri, M. Gilli, "On Stability of Cellular Neural Networks," *Journal of VLSI Signal Processing*, Kluwer Academic Publisher, vol. 23, no. 2/3, pp. 429-435, 1999.

[7] F.Zou, J.A.Nossek, "Stability of Cellular Neural Networks with Opposite-Sign Templates," *IEEE Trans. Circuits and Syst.*, vol. 38, no. 6, pp. 675-677, June 1991.

[8] F.Zou, J.A.Nossek, "Bifurcation and Chaos in Cellular Neural Networks," *IEEE Trans. Circuits and Syst.-I*, vol. 40, no. 3, pp. 166-173, March 1993.

[9] M. Biey, M. Gilli, P. Checco, "Complex dynamic phenomena in space-invariant cellular neural networks," *IEEE Trans. Circuits and Syst.-I*, vol. 49, no. 3, pp. 340-344, March 2002.

[10] I. Petras, T. Roska, L. O. Chua, "New spatial-temporal patterns and the first programmable on-chip bifurcation test-bed," *7-th IEEE International Workshop on Cellular Neural Networks and their Applications*, Frankfurt, July 2002.

[11] P. Thiran, "Influence of Boundary Conditions on the Behavior of Cellular Neural Networks," *IEEE Trans. Circuits and Syst.-I*, vol. 40, no. 3, pp. 207-212, March 1993.

[12] M. Gilli, "Stability of Cellular Neural Networks and Delayed Cellular Neural Networks with Non-positive Templates and Non-monotonic Output Functions," *IEEE Trans. Circuits and Syst.-I*, vol. 41, no. 8, pp. 518-528, August 1994.

Equilibrium points	$Y_{N+1} < \frac{-(p-1+s)}{s}$	$\lvert Y_{N+1}+1\rvert < \frac{p-1}{s}$	$\lvert Y_{N+1}\rvert < \frac{-(p-1)+s}{s}$
$Y_0 < \frac{-(p-1+s)}{s}$	$-1,\{-1\}^n,+1,-1,\{+1,-1\}^m,1$ $-1,\{-1\}^n,+1$	$-1,\{-1\}^n,+1,-1,\{+1,-1\}^m,1$ $-1,\{-1\}^n,1$ $-1,\{-1\}^n,-1$	$-1,\{-1\}^n,-1$
$\lvert Y_0+1\rvert < \frac{p-1}{s}$	$-1,\{-1\}^n,+1,-1,\{+1,-1\}^m,1$ $+1,-1,\{+1,-1\}^n,+1$ $-1,\{-1\}^n,+1$	$-1,\{-1\}^n,+1,-1,\{+1,-1\}^m,1$ $+1,-1,\{+1,-1\}^n,+1$ $-1,\{-1\}^n,+1$ $-1,\{-1\}^n,-1$	$-1,\{-1\}^n,-1$
$\lvert Y_0\rvert < \frac{-(p-1)+s}{s}$	$-1,+1,-1,\{+1,-1\}^m,+1$ $+1,-1,\{+1,-1\}^m,+1$	$-1,+1,-1,\{+1,-1\}^n,+1$ $+1,-1,\{+1,-1\}^n,+1$	No Equilibrium Points
$\lvert Y_0-1\rvert < \frac{p-1}{s}$	$+1,\{+1\}^n,-1,\{+1,-1\}^m,+1$ $-1,+1,-1,\{+1,-1\}^n,+1$ $+1,\{+1\}^n,+1$	$+1,\{+1\}^n,-1,\{+1,-1\}^m,+1$ $-1,+1,-1,\{+1,-1\}^n,+1$ $+1,\{+1\}^n,+1$	$+1,\{+1\}^n,+1$
$Y_0 > \frac{p-1+s}{s}$	$+1,\{+1\}^n,-1,\{+1,-1\}^m,+1$ $+1,\{+1\}^n,+1$	$+1,\{+1\}^n,-1,\{+1,-1\}^m,+1$ $+1,\{+1\}^n,+1$	$+1,\{+1\}^n,+1$

Equilibrium points:	$\lvert Y_{N+1}-1\rvert < \frac{p-1}{s}$	$Y_{N+1} > \frac{p-1+s}{s}$
$Y_0 < \frac{-(p-1+s)}{s}$	$-1,\{-1\}^n,+1,\{-1,+1\}^m,-1$ $-1,\{-1\}^n,-1$	$-1,\{-1\}^n,+1,\{-1,+1\}^m,-1$ $-1,\{-1\}^n,-1$
$\lvert Y_0+1\rvert < \frac{p-1}{s}$	$1,-1,+1,\{-1,+1\}^n,-1$ $-1,+1,\{-1,+1\}^n,-1$	$1,-1,+1,\{-1,+1\}^n,-1$ $-1,+1,\{-1,+1\}^n,-1$
$\lvert Y_0\rvert < \frac{-(p-1)+s}{s}$	$-1,\{-1\}^n,+1,\{-1,+1\}^m,-1$ $-1,\{-1\}^n,-1$	$-1,\{-1\}^n,+1,\{-1,+1\}^m,-1$ $-1,\{-1\}^n,-1$
$\lvert Y_0-1\rvert < \frac{p-1}{s}$	$+1,\{+1\}^n,-1,+1,\{-1,+1\}^m,-1$ $-1,+1,\{-1,+1\}^n,-1$ $+1,\{+1\}^n,-1$ $+1,\{+1\}^n,+1$	$+1,\{+1\}^n,-1,+1,\{-1,+1\}^m,-1$ $-1,+1,\{-1,+1\}^n,-1$ $+1,\{+1\}^n,-1$
$Y_0 > \frac{p-1+s}{s}$	$+1,\{+1\}^n,-1,+1,\{-1,+1\}^m,-1$ $+1,\{+1\}^n,-1$ $+1,\{+1\}^n,+1$	$+1,\{+1\}^n,-1,+1,\{-1,+1\}^m,-1$ $+1,\{+1\}^n,-1$

Table 1: *Equilibrium point patterns in a CNN composed by N cells and described by a 1D template $[s,\ p,\ -s]$, with $p-1 < s$ and $s > 0$ and boundary conditions Y_0 and Y_{N+1}. The two parameters n and m are nonnegative integer numbers, with the constrain that the whole length of the string equals the number N of cells. The string expression $\{a,b\}^0$ represent the null string, whereas the expression $\{a,b\}^n$ denotes a string obtained by repeating n times the symbols a and b, e.g. $\{a,b\}^3 = a,b,a,b,a,b$.*