On the Relationship between CNNs and PDEs

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Abstract

The relationship between Cellular Neural/Nonlinear Networks (CNNs) and Partial Differential Equations (PDEs) is investigated. The equivalence between a discrete-space CNN model and a continuous-space PDE model is rigorously defined. The problem of the equivalence is split into two sub-problems: approximation and topological equivalence, that can be explicitly studied for any CNN models. It is known that each PDE can be approximated by a space difference scheme, i.e. a CNN model, that presents a similar dynamic behavior. It is shown, through several examples, that there exist CNN models that are not equivalent to any PDEs, either because they do not approximate any PDE models, or because they have a different dynamic behavior (i.e they are not topologically equivalent to the PDE, that approximate). This proves that the spatio-temporal CNN dynamics is broader than that described by PDEs.

1 Introduction

Cellular Neural/Nonlinear Networks (CNNs) are analog dynamic processors arrays [1]. A CNN can be described as a 2 or n-dimensional array of identical nonlinear dynamical systems (called cells), that are locally interconnected. A stored programmable array computer combining CNN dynamics with logic (analogic array) has been invented [2], keeping the mainly locally connected property. The latter property has allowed the realization of several high-speed VLSI chips [3]. In most applications the connections are specified through space-invariant templates (that consist of small sets of parameters identical for all the cells.) The mathematical model of a CNN consists of a large set of coupled nonlinear ordinary differential equations (ODEs), that may exhibit a rich spatio-temporal dynamics.

Partial Differential Equations (PDEs) are well known models for describing many classical spatio-temporal phenomena, occurring in physics, chemistry and biology [4].

The relationship between CNNs and PDEs was investigated in several papers: in [5] it was shown that CNNs are a paradigm for several spatio-temporal phenomena, occurring in reaction diffusion PDEs; in [6] some basic methods for simulating linear and nonlinear PDEs, through CNNs, are introduced and some significant examples are given; in [7] it is shown that circuit models are suitable for simulating nonlinear fluid dynamic equations, because they can preserve the physical properties of the continuous structure; in [8] the dynamic behavior of 1D CNNs is examined in detail, with reference to the properties of the corresponding continuous PDEs.

It is known that each PDE can be approximated by space difference schemes, that present a similar dynamic behavior: such schemes can be interpreted as suitable CNNs described by ODEs [6, 9, 10]. On the other hand it is shown in [11] that there are some spatio-temporal phenomena, like propagation failure, that can only be observed in spatially discrete structures and not in their continuous counterpart.

It is therefore important to investigate in a rigorous way the relationship between CNNs and PDEs, in order to determine which are the conditions under which the dynamic behavior of a CNN is similar (i.e. equivalent) to that of a given PDE. Such a study, that has not been carried out in the above mentioned papers [5]-[8], is essential for establishing if we may expect that CNN dynamics is broader than PDE dynamics.

In this paper we firstly identify the mathematical model of a substantial class of CNNs, that we call canonical CNN equations. We rigorously define the notion of equivalence between a canonical CNN equation and a PDE model. We show that the equivalence problem can be split into two sub-problems: approximation and topological equivalence. We develop a general technique, based on Taylor series expansion, for verifying that a canonical CNN equation approximates a PDE model; then we show that the study of the topological equivalence can be carried out through bifurcation analysis, by assuming as bifurcation parameters the space discretization steps. Finally through some significant examples, we show that there exist canonical CNN equations that are not equivalent to any PDEs, either because they do not approximate any PDE models, or because they are not topologically equivalent to the PDE that approximate. This proves that the CNN spatio-temporal dynamics is broader than PDE dynamics.

2 CNN and PDE models

The mathematical model of a canonical CNN equation, formally named Cellular Partial Difference Differential Equations (CPPDE), can be synthesized as reported in the following definition.

Definition 1 : A CPPDE model is a system of $N \times M$ nonlinear ODEs:

$$L(D_t)x_{ij}(t) = \sum_{(k,l)\in N_r(i,j)} T^A_{ij,kl}(x_{ij}, x_{kl}) f(x_{kl}) + \sum_{(k,l)\in N_r(i,j)} T^B_{ij,kl}(u_{ij}, u_{kl}) u_{kl} + \sum_{(k,l)\in N_r(i,j)} T^C_{ij,kl}(x_{ij}, x_{kl}) x_{kl} + z_{ij} \quad (1)$$

The state variables x_{ij} are assumed to be arranged on a regular rectangular grid and are denoted by two indexes $(1 \le i \le N,\ 1 \le j \le M)$; $f(\cdot)$ is a $C^\infty(R)$ nonlinear function and represents the output; u_{ij} is the input; z_{ij} is a constant bias term; $L(D_t)$ is a polynomial function of the differential operator $D_t = d/dt$; $T^A_{ij,kl}(x_{ij}, x_{kl})$ and $T^C_{ij,kl}(x_{ij}, x_{kl})$ are the output and the state feedback templates, that in general might be space-variant nonlinear functions of the state variables x_{ij} and x_{kl}; $T^B_{ij,kl}(u_{ij}, u_{kl})$ is the input template. $N_r(i,j)$ denotes the neighborhood of interaction of each state-variable x_{ij}. The model is completed by specifying the *initial conditions*, i.e. $x_{ij}(0)$ and the space *boundary conditions*.

For the sake of the simplicity, hereafter we assume that the boundary conditions be null or zero-flux and that the templates above T^A and T^C be linear and space-invariant; the inputs u_{ij} and the constant z_{ij} are assumed to be null as well. Under such assumptions, equations (1) with the associated initial conditions yield:

$$\begin{aligned} L(D_t)x_{ij}(t) &= \sum_{|n|\le r, |m|\le r} T^A_{mn}\, f(x_{i+m,j+n}) + \sum_{|n|\le r, |m|\le r} T^C_{mn}\, x_{i+m,j+n} \quad (1 \le i \le N,\ \ 1 \le j \le M) \\ x_{i,j}(0) &= x_{0ij} \qquad (1 \le i \le N,\ \ 1 \le j \le M) \end{aligned} \quad (2)$$

where r denotes the neighborhood radius and x_{0ij} denote arbitrary initial conditions.

Definition 2 : We introduce the set of associated PDEs that formally corresponds to equations (2). The two space variables are denoted with z and w and are defined on a rectangular domain $(0 \le z \le l_z;\ \ 0 \le w \le l_w)$. The equations (with the associated initial conditions) are reported below:

$$\begin{aligned} L(D_t)\tilde{x}(z,w,t) &= L_A(D_z, D_w)\, f[\tilde{x}(z,w,t)] + L_C(D_z, D_w)\, \tilde{x}(z,w,t) \qquad (0 \le z \le l_z;\ \ 0 \le w \le l_w) \\ \tilde{x}(z,w,0) &= \tilde{x}_0(z,w) \qquad (0 \le z \le l_z;\ \ 0 \le w \le l_w) \end{aligned} \quad (3)$$

where $\tilde{x}(z,w,t)$ is assumed to be a C^∞ function of the three variables z, w and t; $L_A(D_z, D_w)$ and $L_C(D_z, D_w)$ are non-constant polynomial functions of the two space differential operators $D_z = \partial/\partial z$ and $D_w = \partial/\partial w$; $\tilde{x}_0(z,w)$ is a C^∞ function of the two space variables z and w; $f(\cdot)$ is the C^∞ function defined in (1).

Definition 3 : The vector $\overline{x}_{ij}(t)$ is defined as follows:

$$\overline{x}_{ij}(t) = \tilde{x}(i\, h_z, j\, h_w,\, t) \qquad h_z = \frac{l_z}{N};\ \ h_w = \frac{l_w}{M};\ \ (1 \le i \le N,\ \ 1 \le j \le M) \quad (4)$$

where h_z and h_w are the two space discretization steps.

In order to study the relationship between CPDDE and PDE models the CPDDE model (2) should describe space-time phenomenon, occurring in the same domain in which the PDE model is defined, i.e. $(0 \le z \le l_z;\ \ 0 \le w \le l_w)$. Therefore the templates T^A and T^C should depend on the space discretization steps $h_z = l_z/N$ and $h_w = l_w/M$; we assume that they are polynomial functions of the variables $1/h_z$ and $1/h_w$ (if this is not the case we assume that they can be approximated to any accuracy through a Taylor polynomial).

Definition 4 : The norm of a vector defined on a rectangular grid, $\boldsymbol{v} = \{v_{ij},\ (1 \le i \le N;\ 1 \le j \le M)\}$ is defined as:

$$||\boldsymbol{v}||_g = \sup_{i,j}|v_{ij}| \quad (5)$$

Definition 5 : A CPDDE model described by equations (2) approximates a PDE model described by equations (3) if and only if there exist two space differential operators $L_A(D_z, D_w)$ and $L_C(D_z, D_w)$ such that

$$\forall t : \quad \lim_{(h_z,h_w)\to(0,0)} ||\boldsymbol{\Delta}(t)||_g = 0 \tag{6}$$

where

$$\begin{aligned}
\boldsymbol{\Delta}(t) &= \{\Delta_{ij}(t) = \Gamma_{ij}(t) - \Psi_{ij}(t),\ (1 \le i \le N;\ 1 \le j \le M)\} \\
\Gamma_{ij}(t) &= \sum_{|n|\le r, |m|\le r} T^A_{mn}\, f\{\tilde{x}[(i+n)\,h_z,\ (j+m)\,h_w,\ t]\} + \sum_{|n|\le r, |m|\le r} T^C_{mn}\, \tilde{x}[(i+n)\,h_z,\ (j+m)\,h_w,\ t] \\
\Psi_{ij}(t) &= L_A(D_z, D_w)\, f\{\tilde{x}[i\,h_z,\ j\,h_w,\ t]\} + L_C(D_z, D_w)\, \tilde{x}[i\,h_z,\ j\,h_w,\ t]
\end{aligned} \tag{7}$$

We remark that if for $(h_z,\ h_w) \rightarrow (0,0)$ the two polynomials L_A and L_C reduce to constants, then the corresponding PDE (3) reduces to a set of uncoupled ODEs.

Definition 6 : A CPDDE model, described by (2) is *topologically equivalent* to a PDE model, described by (3), for given space discretization steps h_z and h_w, if there exists a C^0-diffeomorphism $g(\cdot)$ such that

$$g[\boldsymbol{\Phi}_{t_1}(\overline{\boldsymbol{x}}_0)] = \overline{\boldsymbol{\Phi}}_{t_2}[g(\overline{\boldsymbol{x}}_0)] \tag{8}$$

where $\overline{\boldsymbol{x}}_0 = \{\overline{x}_{ij}(0),\ (1 \le i \le N;\ 1 \le j \le M)\}$; $\boldsymbol{\Phi}_{t_1}$ denotes the trajectory $\boldsymbol{x}(t_1) = \{x_{ij}(t_1),\ (1 \le i \le N;\ 1 \le j \le M)\}$ of the system (2), at time instant $t = t_1$, starting from the initial condition $\overline{\boldsymbol{x}}_0$; $\overline{\boldsymbol{\Phi}}_{t_2}$ denotes the trajectory $\overline{\boldsymbol{x}}(t_2) = \{\overline{x}_{ij}(t_2),\ (1 \le i \le N;\ 1 \le j \le M)\}$, of the discretized system defined in (4), at time instant $t = t_2$ and starting from the initial condition $g(\overline{\boldsymbol{x}}_0)$.

Definition 6 implies that the two systems (2) and (3) should exhibit the following properties, for given space discretization steps h_z and h_w:

- If the nonlinear PDE defined by (3) admits of a unique stable steady-state solution (attractor) for all the possible initial conditions $\tilde{x}(z, w, 0) = \tilde{x}_0(z, w)$, then the corresponding CPDDE (2) presents a single stable attractor for all the possible initial conditions $x_{ij}(0)$
- If the nonlinear PDE defined by (3) exhibits the following set of S stable steady-state solutions (attractors) $\overline{\mathcal{A}} = \{\overline{\mathcal{A}}_1, ..., \overline{\mathcal{A}}_S\}$, then the CPDDE model (2) should present a set of attractors $\mathcal{A}$ that is in a one-to-one correspondence with $\overline{\mathcal{A}}$, i.e $\mathcal{A} = \{\mathcal{A}_1, ..., \mathcal{A}_S\}$.

The above consideration suggests the following procedure for verifying the *topological equivalence*:

Algorithm 1 : Let (2) be a CPDDE model, described by the template $T^A(h_z, h_w)$ and $T^C(h_z, h_w)$, that approximates the PDE model (3) for $(h_z, h_w) \rightarrow (0,0)$. For finite values of the space discretization steps $h_z = \overline{h}_z$ and $h_w = \overline{h}_w$, the *topological equivalence* between the two models can be verified according to the following two steps:

1. Check that there exists $\varepsilon > 0$ such that the CPDDE system (2) does not exhibit any bifurcation phenomena for $0 < h_z < \varepsilon$, $0 < h_w < \varepsilon$.
2. Check that the CPDDE system (2) does not present bifurcations for $0 < h_z < \overline{h}_z + \varepsilon$, $0 < h_w < \overline{h}_w + \varepsilon$.

Note that the above points require to verify that for a given range of the parameters h_z and h_w the CPDDE system does not present either local or global bifurcations. The local bifurcation analysis can be carried out by examining the invariant limit sets of the system for $(h_z,\ h_w) \rightarrow 0$ (i.e. equilibrium points, limit cycles, non-periodic attractors); hence by studying their local properties, i.e. the equilibrium point eigenvalues and the limit cycle Floquet's multipliers. The global bifurcation analysis is more difficult and could become a formidable task because it would require to determine the stable and the unstable manifolds of the invariant limit sets. The bifurcation analysis for the case of the discrete Nagumo-equation is developed in Example 2 of Section 4.

Definition 7 : A CPDDE model, described by the template $T^A(h_z, h_w)$ and $T^C(h_z, h_w)$ is *equivalent* to a PDE model, for finite values of $h_z = \overline{h}_z$ and $h_w = \overline{h}_w$, if and only if the CPDDE model approximates the PDE model for $(h_z, h_w) \rightarrow (0,0)$ and the two models are *topologically equivalent* for $h_z = \overline{h}_z$ and $h_w = \overline{h}_w$.

It is derived that if a CPPDE and a PDE model are equivalent, they presents a similar dynamic behavior and exhibits the same qualitative spatio-temporal phenomena. If this is not the case, the two models in general presents a different dynamics.

3 Examples

In this section we consider some CPDDE models and we investigate the conditions that guarantee the equivalence to a PDE model, according to Definition 7. The equivalence will be studied by verifying that the CPDDE model approximates the PDE model (see Def. 5) and that the two model are topologically equivalent (see Def. 6).

Example 1 : Let us consider a linear CPDDE model, described by the following one-dimensional templates:

$$\begin{aligned} T^A &= 0 \\ T^C &= [r,\ p,\ s] \end{aligned} \tag{9}$$

In order to study the relationship between PDE and CPDDE models, the template elements should be expressed as a function of the space-step h_z. We assume that such functions can be expanded in a Taylor series of the inverse of the space-step h_z and that the series can be truncated at a suitable order. Hence the template elements are written as a polynomial function of the inverse of the space-step h_z (that, hereafter, for the sake of simplicity, will be denoted with h):

$$\begin{aligned} p &= p_0 + \frac{p_1}{h} + \frac{p_2}{h^2} + \frac{p_3}{h^3} + \ldots + \frac{p_L}{h^L} \\ s &= s_0 + \frac{s_1}{h} + \frac{s_2}{h^2} + \frac{s_3}{h^3} + \ldots + \frac{s_L}{h^L} \\ r &= r_0 + \frac{r_1}{h} + \frac{r_2}{h^2} + \frac{r_3}{h^3} + \ldots + \frac{r_L}{h^L} \end{aligned} \tag{10}$$

Such templates give rise to the following CPDDE model:

$$L(D_t)x_i(t) = p\, x_i(t) + s\, x_{i+1}(t) + r\, x_{i-1}(t) \qquad (1 \le i \le N) \tag{11}$$

According to Def. 7, the equivalence between the above CPDDE model (11) and the corresponding PDE models is split in two sub-problems: the identification of the PDE models that are approximated by (11), according to Def. 5; the study of the topological equivalence, as defined in Def. 6.

Approximation: We present a general technique for investigating the approximation problem, that is valid for all the CPDDE models. Def. 5 requires to compute $\tilde{x}[(i+1)h]$ and $\tilde{x}[(i-1)h]$, that can be expressed through the following Taylor series expansion (where $z_i = h\, i$):

$$\begin{aligned} \tilde{x}(z_i + h, t) &= \tilde{x}(z_i, t) + \frac{\partial \tilde{x}}{\partial z}(z_i, t)\, h + \frac{\partial^2 \tilde{x}}{\partial z^2}(z_i, t)\, \frac{h^2}{2} + \frac{\partial^3 \tilde{x}}{\partial z^3}(z_i, t)\, \frac{h^3}{6} + \ldots + \frac{\partial^K \tilde{x}}{\partial z^K}(\alpha_i, t)\, \frac{h^K}{K!} \\ &\qquad \alpha_i \in [z_i,\ z_i + h] \\ \tilde{x}(z_i - h, t) &= \tilde{x}(z_i, t) - \frac{\partial \tilde{x}}{\partial z}(z_i, t)\, h + \frac{\partial^2 \tilde{x}}{\partial z^2}(z_i, t)\, \frac{h^2}{2} - \frac{\partial^3 \tilde{x}}{\partial z^3}(z_i, t)\, \frac{h^3}{6} + \ldots + \frac{\partial^K \tilde{x}}{\partial z^K}(\beta_i, t)\, \frac{(-h)^K}{K!} \\ &\qquad \beta_i \in [z_i - h,\ z_i] \end{aligned} \tag{12}$$

In order to evaluate $\Delta_i(t)$ (see Def. 5) one has to compute $\Gamma_i(t)$ and then to determine (if there exists) a differential space operator $L_C(D_z)$ such that (6) is satisfied. The quantity $\Gamma_i(t)$ can be computed by using the series expansion (12). We have:

$$\begin{aligned} \Gamma_i(t) &= p\, \tilde{x}(z_i, t) + s\, \tilde{x}(z_i + h, t) + r\, \tilde{x}(z_i - h, t) \\ &= (p + s + r)\, \tilde{x}(z_i, t) + (s - r)\, h\, \frac{\partial \tilde{x}}{\partial z}(z_i, t) + (s + r)\, \frac{h^2}{2} \frac{\partial^2 \tilde{x}}{\partial z^2}(z_i, t) + (s - r)\, \frac{h^3}{6} \frac{\partial^3 \tilde{x}}{\partial z^3}(z_i, t) + \ldots \\ &\quad \ldots + \frac{h^K}{K!} \left(s \frac{\partial^K \tilde{x}}{\partial z^K}(\alpha_i, t) + (-1)^K r \frac{\partial^K \tilde{x}}{\partial z^K}(\beta_i, t) \right) \end{aligned} \tag{13}$$

The terms of the above expression (13) are finite, as $h \to 0$, if and only if the template element coefficients satisfy the following conditions:

$$\begin{aligned} p_1 + s_1 + r_1 &= 0 & \qquad p_2 + s_2 + r_2 &= 0 \\ s_2 - r_2 &= 0 & \qquad p_k = s_k = r_k &= 0 \quad (k \ge 3) \end{aligned} \tag{14}$$

Therefore $\Gamma_i(t)$, by use of (13), (14) and of (10), can be written as:

$$\begin{aligned}\Gamma_i(t) &= (p_0+s_0+r_0)\,\tilde{x}(z_i,t)+(s_0-r_0)\,h\,\frac{\partial\tilde{x}}{\partial z}(z_i,t)+(s_1-r_1)\,\frac{\partial\tilde{x}}{\partial z}(z_i,t)\\ &+(s_0+r_0)\,\frac{h^2}{2}\,\frac{\partial^2\tilde{x}}{\partial z^2}(z_i,t)+\frac{s_1+r_1}{2}\,h\,\frac{\partial^2\tilde{x}}{\partial z^2}(z_i,t)+\frac{s_2+r_2}{2}\,\frac{\partial^2\tilde{x}}{\partial z^2}(z_i,t)\\ &+\frac{h^3}{6}\left[\left(s_0+\frac{s_1}{h}+\frac{s_2}{h^2}\right)\frac{\partial^3\tilde{x}}{\partial z^3}(\alpha_i,t)-\left(r_0+\frac{r_1}{h}+\frac{r_2}{h^2}\right)\frac{\partial^3\tilde{x}}{\partial z^3}(\beta_i,t)\right]\end{aligned} \tag{15}$$

The above expression allows to identify the space differential operator $L_C(D_z)$. We have:

$$L_C(D_z)[\tilde{x}(z,t)]=(p_0+s_0+r_0)\,\tilde{x}(z,t)+(s_1-r_1)\,\frac{\partial\tilde{x}}{\partial z}(z,t)+\frac{s_2+r_2}{2}\,\frac{\partial^2\tilde{x}}{\partial z^2}(z,t) \tag{16}$$

By use of (15) and of (16), the quantity $\Delta_i(t)$, reported in Def. 5, is readily computed as:

$$\begin{aligned}\Delta_i(t) &= (s_0-r_0)\,h\,\frac{\partial\tilde{x}}{\partial z}(z_i,t)+(s_0+r_0)\,\frac{h^2}{2}\,\frac{\partial^2\tilde{x}}{\partial z^2}(z_i,t)+\frac{s_1+r_1}{2}\,h\,\frac{\partial^2\tilde{x}}{\partial z^2}(z_i,t)\\ &+\frac{h^3}{6}\left[\left(s_0+\frac{s_1}{h}+\frac{s_2}{h^2}\right)\frac{\partial^3\tilde{x}}{\partial z^3}(\alpha_i,t)-\left(r_0+\frac{r_1}{h}+\frac{r_2}{h^2}\right)\frac{\partial^3\tilde{x}}{\partial z^3}(\beta_i,t)\right]\end{aligned} \tag{17}$$

Since the function $\tilde{x}(z,t)$ has been assumed to be C^∞, its derivatives are bounded for each z_i; hence we derive:

$$\forall t,i:\qquad \lim_{h\to 0}|\Delta_i(t)|=0, \tag{18}$$

that according to Def. 4 and 5 yields:

$$\forall t:\qquad \lim_{h\to 0}||\boldsymbol{\Delta}(t)||_g=0 \tag{19}$$

The following considerations hold:

1. The result above (19) shows that if the conditions (14) are matched then the CPDDE model (11) **approximates** the following PDE for $h\to 0$:

   $$L(D_t)[\tilde{x}(z,t)]=(p_0+s_0+r_0)\,\tilde{x}(z,t)+(s_1-r_1)\,\frac{\partial\tilde{x}}{\partial z}(z,t)+\frac{s_2+r_2}{2}\,\frac{\partial^2\tilde{x}}{\partial z^2}(z,t) \tag{20}$$

   We note that, if the space differential operator $L_C(D_z)$ is a constant polynomial (for instance if $p=p_0$, $s=s_0$, $r=r_0$), the CPPDE model reduces to a set of uncoupled ODEs, that is of no interest in this study.

2. If the conditions (14) are not satisfied, then the CPDDE model (11) **does not approximate** any PDE, for $h\to 0$.

Topological equivalence: Let us assume that the conditions (14) are fulfilled, i.e. the CPDDE model (11) *approximates* the PDE model (20), according to Def. 5. An autonomous linear system may present only two dynamic behaviors (with the exception of a set of parameters of measure zero): a) stability, which implies the existence of a single attractor (globally stable equilibrium point); b) instability, in case each trajectory (with the exception of a set of measure zero) diverges. In the following we assume that the linear PDE model (20) be stable.

According to Def. 6, the *topological equivalence* is defined for a finite value of the space discretization step h. For stable systems the check of the *topological equivalence* simply require to verify that the ODE, described by (11) be stable, i.e. all the N eigenvalues have negative real part.

In the particular case $L(D_t)=\partial/\partial t$, the explicit computation of the eigenvalues yields the following stability conditions

$$\boldsymbol{Re}[\lambda_i]=\boldsymbol{Re}[p+2\sqrt{rs}\cos\left(\frac{i\pi}{N+1}\right)]<0,\qquad (1\le i\le N) \tag{21}$$

where $N=l_z/h$, according to (4).

As a final result we can state that for a given space discretization step h, the CPDDE model (11) is *equivalent* to the PDE model (20) if and only if both the approximation (14) and the stability conditions above (21) are satisfied.

If this is not the case, the CPDDE model (11) is not equivalent to the PDE (20).

Example 2 : Let us consider a nonlinear CPDDE model, described by the following one-dimensional templates:

$$\begin{aligned} T^A &= [r^A \; p^A \; s^A] \\ T^C &= [r^C, \; p^C, \; s^C] \end{aligned} \tag{22}$$

with

$$\begin{aligned} p^{A,C} &= p_0^{A,C} + \frac{p_1^{A,C}}{h} + \frac{p_2^{A,C}}{h^2} + \frac{p_3^{A,C}}{h^3} + \ldots + \frac{p_L^{A,C}}{h^L} \\ s^{A,C} &= s_0^{A,C} + \frac{s_1^{A,C}}{h} + \frac{s_2^{A,C}}{h^2} + \frac{s_3^{A,C}}{h^3} + \ldots + \frac{s_L^{A,C}}{h^L} \\ r^{A,C} &= r_0^{A,C} + \frac{r_1^{A,C}}{h} + \frac{r_2^{A,C}}{h^2} + \frac{r_3^{A,C}}{h^3} + \ldots + \frac{r_L^{A,C}}{h^L} \end{aligned} \tag{23}$$

The following CPDDE model is obtained:

$$L(D_t)x_i(t) = p^A \, f[x_i(t)] + s^A \, f[x_{i+1}(t)] + r^A \, f[x_{i-1}(t)] + p^C \, x_i(t) + s^C \, x_{i+1}(t) + r^C \, x_{i-1}(t) \quad (1 \le i \le N) \tag{24}$$

Approximation: Since the nonlinear function $f(\cdot)$ has been assumed to be C^∞, it admits of a Taylor expansion. By proceeding as in the Example 1, we find that the CPDDE model (24) approximates a PDE if and only if the following constraints are fulfilled:

$$\begin{aligned} p_1^{A,C} + s_1^{A,C} + r_1^{A,C} &= 0 & \qquad p_2^{A,C} + s_2^{A,C} + r_2^{A,C} &= 0 \\ s_2^{A,C} - r_2^{A,C} &= 0 & \qquad p_k^{A,C} = s_k^{A,C} = r_k^{A,C} &= 0 \quad (k \ge 3) \end{aligned} \tag{25}$$

The corresponding PDE is:

$$\begin{aligned} L(D_t)[\tilde{x}(z,t)] = {} & (p_0^A + s_0^A + r_0^A)\, f[\tilde{x}(z,t)] + (s_1^A - r_1^A)\, \frac{\partial f[\tilde{x}(z,t)]}{\partial z} + \frac{s_2^A + r_2^A}{2}\, \frac{\partial^2 f[\tilde{x}(z,t)]}{\partial z^2} \\ & + (p_0^C + s_0^C + r_0^C)\, \tilde{x}(z,t) + (s_1^C - r_1^C)\, \frac{\partial \tilde{x}}{\partial z}(z,t) + \frac{s_2^C + r_2^C}{2}\, \frac{\partial^2 \tilde{x}}{\partial z^2}\,(z,t) \end{aligned} \tag{26}$$

Topological equivalence: Since the CPDDE model (24) is nonlinear, the study of the *topological equivalence*, according to Def. 6 and Algorithm 1 is more complex.

We restrict our attention to the well known case of the Nagumo equations, studied by Keener in [12]:

$$\frac{\partial \tilde{x}}{\partial t} = D \frac{\partial^2 \tilde{x}}{\partial z^2} + f(\tilde{x}) \tag{27}$$

where D is the diffusion coefficient. We assume zero-flux boundary conditions, i.e.

$$\frac{\partial \tilde{x}}{\partial z}\,|_{z=0,l} \, = 0 \tag{28}$$

Such an equation corresponds to the PDE and the CPDDE models (26) and (24) respectively, by assuming $L(D_t) = D_t$ and, in addition to (25), the following constraints:

$$\begin{aligned} p_0^A + s_0^A + r_0^A &= 1 & \qquad s_1^A = s_2^A = r_1^A = r_2^A &= 0 \\ p_0^C + s_0^C + r_0^C &= 0 & \qquad s_1^C = r_1^C &= 0 \\ \frac{s_2^C + r_2^C}{2} &= D & & \end{aligned} \tag{29}$$

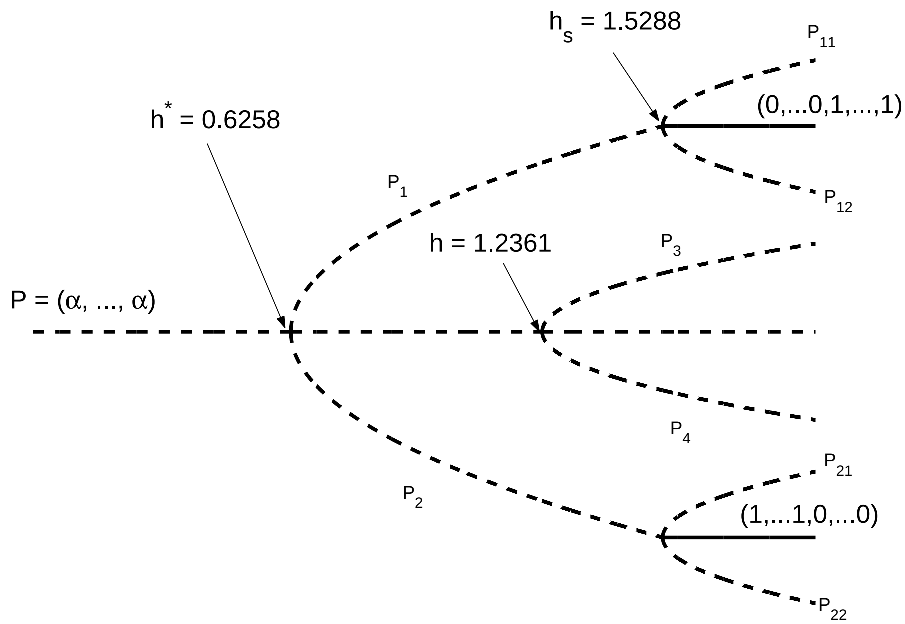

Figure 1: Equilibrium point bifurcation for the 10 cell CPDDE model (31)

We also suppose that the nonlinear function $f(\cdot)$ can be approximated by the following expression (for $0 \leq \alpha \leq 0.5$).

$$f(\tilde{x}) = \begin{cases} -\alpha\tilde{x} & \tilde{x} \leq 0 \\ -\tilde{x}^3 + (1+\alpha)\tilde{x}^2 - \alpha\tilde{x} & 0 \leq \tilde{x} \leq 1 \\ (1-\alpha)(\tilde{x}-1) & \tilde{x} \geq 1 \end{cases} \tag{30}$$

The above function is continuous with its first-order derivative; it coincides with the cubic function considered in [11] for $0 \leq \tilde{x} \leq 1$, whereas it is linear for $|\tilde{x}| > 1$. The latter property allows to simplify the numerical computation of the bifurcation processes occurring in the corresponding CPDDE model. For sake of completeness such a model, with the corresponding zero-flux boundary conditions, is reported below:

$$\begin{aligned} \frac{dx_1(t)}{dt} &= \frac{D}{h^2}\left[x_2(t) - x_1(t)\right] + f[x_1(t)] \\ \frac{dx_i(t)}{dt} &= \frac{D}{h^2}\left[x_{i+1}(t) - 2x_i(t) + x_{i-1}(t)\right] + f[x_i(t)] \qquad (2 \leq i \leq N-1) \\ \frac{dx_N(t)}{dt} &= \frac{D}{h^2}\left[x_{N-1}(t) - x_N(t)\right] + f[x_N(t)] \end{aligned} \tag{31}$$

The analysis of the dynamic system described by (31) yields the following results:

1. For any number N of cells and diffusion coefficient D there exists h^* such that for $0 \leq h < h^*$ the CPDDE system (31) presents only three equilibrium points:

   (a) two stable equilibrium points $P_a = (0, 0, ..., 0, 0)$ and $P_b = (1, 1, ..., 1, 1)$, whose Jacobian matrix exhibits N negative real eigenvalue;

   (b) one unstable equilibrium point $P = (\alpha, \alpha, ..., \alpha, \alpha)$ that presents N positive real eigenvalues.

   This implies, according to Algorithm 1, that for each finite values of $h < h^*$ the CPDDE (31) and the PDE (27) models are *topologically equivalent*. Since we have already shown that (31) approximates (27), then for $h \leq h^*$ the two models are *equivalent*, according to Def. 7.

2. If $h \geq h^*$, then the system (31) still presents the two stable equilibrium points P_a and P_b. The unstable point $P = (\alpha, \alpha, ..., \alpha, \alpha)$ undergoes a series of pitchfork bifurcations, that finally gives rise to the emergence of a number of additional stable equilibrium points. The bifurcation process is studied in detail for a 10

cell network and is reported in Fig. 1: we have verified that the main characteristics are not influenced by the number of cells. It can be described as follows:

(a) For $h < h^* = 0.6258$ the equilibrium point $P = (\alpha, ..., \alpha)$ is a saddle point of index one: its index is denoted with I_P. Then the point P undergoes a series of pitchfork bifurcation. The effect of each bifurcation is to increase by one the index I_P and to create two additional saddles of index I_P. The first two bifurcations, reported in Fig. 2, occurs for $h = h^* = 0.6258$ and $h = 1.2361$; the emerging saddles are denoted with P_1, P_2, P_3, and P_4 respectively.

(b) For $h = h_s = 1.5288$. i.e. after the second bifurcation of P, the two saddles P_1 and P_2 undergoes a pitchfork bifurcation. As a result they become stable nodes and gives birth to four saddles of index one (denoted as P_{11}, P_{12} and P_{21}, P_{22} respectively.) By increasing h the two stable nodes P_1 and P_2 do not undergo further bifurcations and converge to the equilibrium points $(0,0,0,0,0,1,1,1,1,1)$ and $(1,1,1,1,1,0,0,0,0,0)$ of the uncoupled network.

As a result of the above study we can claim that the CPDDE model is not *topologically equivalent* to the PDE model for $h \geq h^*$, i.e. after the occurrence of the first bifurcation. In addition for $h > h_s$ the discrete CPDDE model presents a pair of stable equilibrium pints that are not present in the original PDE, i.e. there is not a one-to-one correspondence between the attractors of the two models.

3. For any number N of cells and diffusion coefficient D, there exists h_f such that for $h > h_f$ the cells can be considered uncoupled, i.e. there is a one-to-one correspondence between the set of equilibrium points of the dynamical system (31) and the set of equilibrium points of N uncoupled cells. Each uncoupled cell x_i presents two stable equilibrium points $P_a^i = 0$ and $P_b^i = 1$ and one unstable point $P^i = \alpha$. It is derived that for $h > h_f$ the whole system possesses 2^N stable equilibrium points (with all negative real eigenvalues) and $3^N - 2^N$ unstable equilibrium points (with at least one positive real eigenvalue).

   In order to show a numerical example of the complete bifurcation process we have considered a CPDDE model composed by 10 cells and we have determined the number of stable equilibrium points as a function of the discretization step h. The results can be summarized as follows (they are reported in Fig. 4)

   (a) For $0 < h < 1.5288$ the CPDDE model exhibits only two globally stable equilibrium points i.e. the origin and the point whose coordinates are all 1. This is in agreement with the detailed bifurcation analysis shown in Fig. 3.

   (b) For $h > 1.5288$ two additional stable equilibrium points emerge; they correspond to the points denoted with P_1 and P_2 in Fig. 2.

   (c) By further increasing h new stable equilibrium points emerges, through pitchfork bifurcations, that are not shown in Fig. 2; finally, for $h > h_f = 5.47$ the cells behave as they were uncoupled, i.e. each cell exhibits two stable equilibrium points, giving rise to a total number of $2^{10} = 1024$ points.

Remark: According to the discussion above and to Def. 6, the propagation failure phenomena, studied in [12], occurs for those $h > h_s$ such that there is not a one-to-one correspondence between the stable equilibrium points of the two models. The statement of [12] regarding the existence of propagation failure for arbitrarily small space discretization steps h, can be reformulated as follows: for each discretization step h (even arbitrarily small) there exists a diffusion coefficients D such that the CPDDE and the PDE model are not *topologically equivalent.*

4 Conclusion

We have investigated the relationship between canonical CNN equations, which are Cellular Partial Difference-Differential Equations (CPDDEs) and PDEs. We have rigorously defined the notion of equivalence between a canonical CNN equation and a PDE and we have shown that such a concept can be split into two problems: approximation and topological equivalence. We have developed a general technique, based on Taylor series expansion, for verifying that a canonical CNN equation approximates a PDE; then we have shown that the study of the topological equivalence can be carried out through bifurcation analysis, by assuming as bifurcation parameters the space discretization steps.

Finally we have reported some significant examples, that show that there exist canonical CNN equations that are not equivalent to any PDE, either because they do not approximate any PDE model, or because they

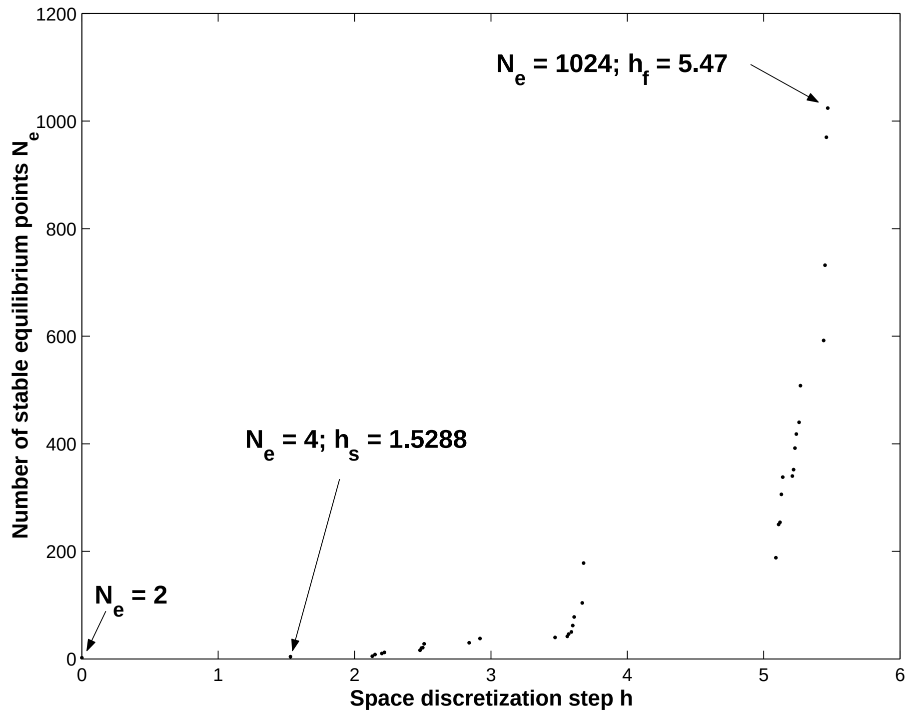

Figure 2: *Number of stable equilibrium points N_e versus the discretization step h for the* 10 *cell CPDDE model (31).*

are not topologically equivalent to the PDE that approximate. This proves that the CNN spatio-temporal dynamics is expected to be broader than the PDE dynamics.

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