Nontrivial blocking sets in PG$(n, 2)$: draft

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Abstract

The smallest nontrivial blocking sets with respect to t-spaces in PG$(n, 2)$ are determined.

1 Introduction

Let PG(n, q) denote the n-dimensional projective space the finite field of order q. A *blocking set with respect to t-spaces in* PG(n, q) is a set of points that has nonempty intersection with every t-space of PG(n, q). Sometimes a blocking set with respect to t-spaces in PG(n, q) is called an *$(n-t)$-blocking set* $\boxed{\text{ref}}$. A blocking set with respect to lines in a projective plane is simply called a *blocking set*.

Theorem 1.1 **(Bose and Burton [5])** *If B is a blocking set with respect to t-spaces in* PG(n, q)*, then $|B| \geq |\mathrm{PG}(n-t, q)|$. Equality holds if and only if B is an $(n-t)$-space.*

A blocking set with respect to t-spaces that contains an $(n-t)$-space is called *trivial*. The smallest nontrivial blocking sets with respect to t-spaces in PG(n, q) are characterised for $q > 2$ in Theorem 1.3.

Theorem 1.2 **(Beutelspacher [1], Heim [7])** *In* PG(n, q)*, $q > 2$, the smallest nontrivial blocking sets with respect to t-spaces, $1 \leq t \leq n-1$, are cones with vertex an $(n-t-2)$-space π_{n-t-2} and base a nontrivial blocking set of minimal cardinality in a plane skew to π_{n-t-2}.*

It is known that if $q > 2$, then PG$(2, q)$ has a nontrivial blocking set and that the size of such a nontrivial blocking set is substantially bigger than $q + 1$, the size of a line.

Theorem 1.3 *Let B be a nontrivial blocking set of* PG$(2, q)$*, $q > 2$.*

1. **(Bruen [6])** *$|B| \geq q + \sqrt{q} + 1$, with equality if and only if B is a Baer subplane.*

2. **(Blokhuis [2])** *If q is a prime, then $|B| \geq 3(q + 1)/2$. This bound is sharp.*

3. **(Blokhuis [3], Blokhuis et al. [4])** *If $q = p^{2e+1}$, p prime, $e \geq 1$, then $|B| \geq \max(q + 1 + p^{e+1}, q + 1 + c_p q^{2/3})$, where c_p equals $2^{-1/3}$ if $p \in \{2, 3\}$ and 1 if $p \geq 5$.*

However, it is not hard to see that if $q = 2$, then every blocking set in $\mathrm{PG}(2, q)$ is trivial. Hence the situation for nontrivial blocking sets with respect to t-spaces in $\mathrm{PG}(n, q)$ must be different from the situation described in Theorem 1.2. In this paper we handle this case and prove the following.

Theorem 1.4 (Check)

1. *In* $\mathrm{PG}(n, 2)$, $n \geq 3$, *the smallest nontrivial blocking sets with respect to hyperplanes are skeletons of a solid in* $\mathrm{PG}(n, 2)$*; these are sets of five points in a 3-space no four of which are coplanar. If* $n = 3$*, then these are the only minimal nontrivial blocking sets with respect to planes.*

2. *Up to isomorphism, there is only one nontrivial minimal blocking set with respect to lines in* $\mathrm{PG}(3, 2)$*. It consists of ten points and can be described as* $l \cup l_1 \cup l_2 \cup l_3$*, where* l_1*,* l_2 *and* l_3 *are three concurrent, not coplanar lines skew to the line* l*.*

3. *In* $\mathrm{PG}(n, 2)$, $n \geq 3$, *the smallest nontrivial blocking sets with respect to t-spaces,* $1 \leq t \leq n - 2$*, have size* $2^{n-t+1} + 2^{n-t-1} + 2^{n-t-2} - 1$ *and are cones with vertex an* $(n - t - 3)$*-space* π_{n-t-3} *and base a nontrivial minimal blocking set with respect to lines in a solid skew to* π_{n-t-3}*.*

In Section 2, the threedimensional case is handled, while Section 3 deals with larger dimensional spaces.

2 In three dimensions

2.1 With respect to planes

Suppose B is a minimal nontrivial blocking set with respect to planes in $\mathrm{PG}(3, 2)$. Let P, Q and R be three points of B and let π be the plane $\langle P, Q, R\rangle$. In π there is a unique line, say l, that contains no point of $\{P, Q, R\}$. It cannot contain a point of B, since otherwise π would contain a line contained in B. Let π' and π'' be the remaining planes through l. They both have to contain a point of B. Let $S \in \pi' \cap B$ and $T \in \pi'' \cap B$. Clearly, $\{P, Q, R, S, T\}$ is a blocking set with respect to planes. Hence $B = \{P, Q, R, S, T\}$. Since B is nontrivial, it contains no lines.

We now show that this implies that no four of its points are coplanar. Assume that B contains a set A consisting of four coplanar points. The set A cannot contain $\{P, Q, R\}$, hence it must contain S, T and two points of $\{P, Q, R\}$, say, without loss of generality, P and Q. Now consider the lines ST and PQ. Since they lie in a plane, they intersect. If they would intersect in the third point of PQ, which is a point of l, then S and T would be contained in the same plane through l, a contradiction. Hence they intersect in either P or Q, implying that the line ST is contained in B, a contradiction.

Hence B consists of five points, no four of which are collinear. Clearly such a set contains no line. Moreover, it is known that such a set is, up to collineations, unique. Is is called a *skeleton* of $\mathrm{PG}(3, 2)$.

2.2 With respect to lines

Suppose B is a minimal nontrivial blocking set with respect to lines in $\mathrm{PG}(3,2)$. If π is a plane, then $B \cap \pi$ is a blocking set in π, such that π contains a line contained in B.

Let P be a point in B, let l be a tangent through P and let π_1, π_2 and π_3 be the three planes through l. Each of these planes must contain a line consisting of points contained in B. These lines must pass through P. Let $l_i \subseteq \pi_i$, $i \in \{1,2,3\}$, be such lines. Let $l'_i := \langle l_j, l_k\rangle \cap \pi_i$ for all i,j,k satifying $\{i,j,k\} = \{1,2,3\}$. Since B contains no plane, the lines l_1, l_2 and l_3 are not coplanar, hence l'_i is the line in π_i through P differen from l and l_i. The lines l'_1, l'_2 and l'_3 are coplanar and the plane $\langle l'_1, l'_2\rangle$ must contain a line l' consisting of points of B. This line cannot equal l'_i for any $1 \leq i \leq 3$, for otherwise the plane $\langle l_j, l_k\rangle$ that contains l'_i would be contained in B. Hence it intersects l'_1, l'_2 and l'_3 in distinct points P'_1, P'_2 and P'_3.

Hence B contains $A := l' \cup (\cup_i l_i)$. We now check that A is a minimal blocking set with respect to lines to conclude that $B = A$ and that it is a nontrivial minimal blocking set with respect to lines in $\mathrm{PG}(n,2)$ of size ten.

To check that A is blocking set it suffices to show that every plane contains a line contained in A. The planes through P are the planes π_1, π_2, π_3, $\langle l_1, l_2\rangle$, $\langle l_2, l_3\rangle$, $\langle l_3, l_1\rangle$ and $\langle P, P'_1, P'_2\rangle$. Clearly, each one of them contains a line contained in A. Now let π be any plane not through P. It intersects l' in a point P'_i for some $i \in \{1,2,3\}$. Let $\{i,j,k\} = \{1,2,3\}$. Then π intersects $\langle l'_i, l_j\rangle = \langle l'_i, l_j, l_k\rangle$ in a line m containing P'_i. Since π does not pass through P, the line m intersects l_j and l_k in distinct points of B. Hence m is contained in B.

It is easy to check that removing a point of A will result in a line skew to the new set. Hence A is a minimal blocking set of size ten.

3 In more dimensions

3.1 With respect to hyperplanes

Suppose B is a minimal notrivial blocking set with respect to hyperplanes in $\mathrm{PG}(n,2)$, $n \geq 4$. As above, consider any three points $\{P,Q,R\}$ and let l the line in $\pi = \langle P,Q,R\rangle$ skew to B. Let S be any point of B outside π and let π_3 be the solid $\langle S, \pi\rangle$. Let π' be the plane $\langle S, l\rangle$ and let π'' be the third plane in π_3 through l. If π'' contains a point of B, then $|B| = 5$ and the reasoning from Subsection 2.1 can be copied to show that B is a skeleton of a solid in $\mathrm{PG}(n,2)$. If π'' contains no points of B, then all hyperplanes of $\mathrm{PG}(n,2)$ containing π'' but not π_3 must contain a point of $B \setminus \pi_3$, implying that B contains at least two points outside π_3, such that $|B| \geq 6$.

3.2 With respect to lines

Suppose that $n \geq 4$, that B is a nontrivial blocking set with respect to lines in $\mathrm{PG}(n,2)$ of size at most $2^n + 2^{n-2} + 2^{n-3} - 1$ and that Theorem 1.3 holds in $\mathrm{PG}(n',2)$ for every $3 \leq n' < n$.

Let $\mathcal{T}$ denote the set of $(n-2)$-spaces contained in B.

Lemma 3.1 *If a hyperplane contains three elements of $\mathcal{T}$, then it contains four. These four $(n-2)$-spaces pass through a common $(n-4)$-space and form a dual hyperoval in the quotient space with respect to this $(n-4)$-space. If there is a hyperplane containing four elements of $\mathcal{T}$, then $|B| \geq 2^n + 2^{n-2} + 2^{n-3} - 1$. A hyperplane cannot contain more than four elements of $\mathcal{T}$.*

Proof $\boxed{\text{Write this down.}}$ □

Lemma 3.2 *Every point of B is contained in at least three elements of $\mathcal{T}$. If a point of B is contained in exactly three elements of $\mathcal{T}$, then $|B| = 2^n + 2^{n-2} + 2^{n-3} - 1$.*

Proof $\boxed{\text{Write this down.}}$ □

Lemma 3.3 *If all elements of $\mathcal{T}$ pas through a commom point, then B is as in Theorem 1.4.*

Proof $\boxed{\text{Write this down.}}$ □

3.3 With respect to t-spaces, $1 < t < n-1$

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References

[1] A. Beutelspacher. Blocking sets and partial spreads in finite projective spaces. *Geom. Dedicata*, 9(4):425–449, 1980.

[2] A. Blokhuis. On the size of a blocking set in $\mathrm{PG}(2,p)$. *Combinatorica*, 14(1):111–114, 1994.

[3] A. Blokhuis. Blocking sets in Desarguesian planes. In *Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993)*, pages 133–155. János Bolyai Math. Soc., Budapest, 1996.

[4] A. Blokhuis, L. Storme, and T. Szőnyi. Lacunary polynomials, multiple blocking sets and Baer subplanes. *J. London Math. Soc. (2)*, 60(2):321–332, 1999.

[5] R. C. Bose and R. C. Burton. A characterization of flat spaces in a finite geometry and the uniqueness of the Hamming and the MacDonald codes. *J. Combin. Theory*, 1:96–104, 1966.

[6] A. A. Bruen. Baer subplanes and blocking sets. *Bull. Amer. Math. Soc.*, 76:342–344, 1970.

[7] U. Heim. Blockierende Mengen in endlichen projektiven Räumen. *Mitt. Math. Sem. Giessen*, (226):82, 1996. Dissertation, Justus-Liebig-Universität Giessen, Giessen, 1995.

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