Partial hemisystems: draft

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Abstract

Introducing a connection to minihypers, we prove extendability results for partial hemisystems.

1 Definitions

Let $\mathrm{PG}(n,q)$ denote the n-dimensional projective space over $\mathrm{GF}(q)$, the finite field of order q. If P is a point of $\mathrm{PG}(n,q)$, then $\mathsf{star}(P)$ denotes the set of lines of $\mathrm{PG}(n,q)$ through P. Let $\mathrm{Q}(4,q)$ denote the nonsingular quadric in $\mathrm{PG}(4,q)$ and $\mathrm{W}_3(q)$ the threedimensional simplectic space over $\mathrm{GF}(q)$.

A *hemisystem* $\mathcal{H}$ on $\mathrm{Q}(4,q)$ is a set of points on $\mathrm{Q}(4,q)$ such that each line of $\mathrm{Q}(4,q)$ contains exactly $(q+1)/2$ points of $\mathcal{H}$. If $\mathcal{H}$ is a hemisystem of $\mathrm{Q}(4,q)$, then $|\mathcal{H}| = (q+1)(q^2+1)/2$. A *partial hemisystem* $\mathcal{H}$ on $\mathrm{Q}(4,q)$ is a set of points on $\mathrm{Q}(4,q)$ such that each line of $\mathrm{Q}(4,q)$ contains at most $(q+1)/2$ points of $\mathcal{H}$. The *deficiency* δ of a partial hemisystem $\mathcal{H}$ of $\mathrm{Q}(4,q)$ is by definition the number of points it lacks to be a hemisystem, whence $\delta = (q+1)(q^2+1)/2 - |\mathcal{H}|$.

Since $\mathrm{Q}(4,q)$ is the point-line dual of $\mathrm{W}_3(q)$, see e.g. [4, §3.2], it makes sense to introduce the dual notion: a *(partial) dual hemisystem* $\mathcal{H}^*$ on $\mathrm{W}_3(q)$ is a set of lines on $\mathrm{W}_3(q)$ such that each point of $\mathrm{W}_3(q)$ is incident with (at most) $(q+1)/2$ lines of $\mathcal{H}^*$. The *deficiency* δ of a partial dual hemisystem equals $(q+1)(q^2+1)/2 - |\mathcal{H}^*|$.

An $\{f,m;n,q\}$-*minihyper* is a pair (F,w), where F is a subset of the point set of $\mathrm{PG}(n,q)$ and w is a weight function $w : \mathrm{PG}(n,q) \to \mathbb{N} : P \mapsto w(P)$, satisfying (i) $w(P) > 0 \Leftrightarrow P \in F$, (ii) $\sum_{P\in F} w(P) = f$, and (iii) $\min\{\sum_{P\in H} w(P) : H \text{ is a hyperplane}\} = m$.

2 The link with minihypers

Theorem 2.1 *Suppose $\mathcal{H}^*$ is a partial dual hemisystem of $\mathrm{W}_3(q)$. Define a weight function w as follows:*

$$w : \mathrm{PG}(3,q) \to \mathbb{N} : P \mapsto \frac{q+1}{2} - |\mathsf{star}(P) \cap \mathcal{H}^*|.$$

If F is the set of points of $\mathrm{PG}(3,q)$ with positive weight, then (F,w) is a $\{\delta(q+1),\delta;3,q\}$-minihyper.

Proof The weight of $\mathrm{PG}(3,q)$ equals

$$\begin{aligned} w(\mathrm{PG}(3,q)) = \sum_{P\in \mathrm{PG}(3,q)} w(P) &= \frac{q+1}{2}(q^3+q^2+q+1) - |\mathcal{H}^*|(q+1) \\ &= \delta(q+1). \end{aligned}$$

A hyperplane π of $\mathrm{PG}(3,q)$ intersects $\mathrm{W}_3(q)$ in a pencil of lines, i.e., the set of lines in a plane that pass through a given point of that plane. Let α denote the number of lines of $\mathcal{H}^*$ contained in π. Clearly, $\alpha \leq (q+1)/2$. So,

$$\begin{aligned} w(\pi) = \sum_{P\in\pi} w(P) &= \frac{q+1}{2}(q^2+q+1) - \alpha(q+1) - (|\mathcal{H}^*| - \alpha) \\ &= \delta + q(\frac{q+1}{2} - \alpha) \ \geq \ \delta. \end{aligned}$$

A theorem of Hamada [2] shows that (F,w) is a $\delta(q+1),\delta;3,q)$-minihyper. In [2] the theorem is proved for minihypers without weight, but the proofs also hold when weights are allowed. $\square$

A *blocking set* in $\mathrm{PG}(2,q)$ is a set of points in $\mathrm{PG}(2,q)$ that meets every line. It is called *trivial* when it contains a line. For information on blocking sets, we refer to [3]. Let $q+\varepsilon_q$ denote the size of the smallest nontrivial blocking sets in $\mathrm{PG}(2,q)$.

Corollary 2.2 *If $\mathcal{H}^*$ is a partial dual hemisystem of $\mathrm{W}_3(q)$ with deficiency $\delta < \varepsilon_q$, then $\mathcal{H}^*$ can be completed to a dual hemisystem of $\mathrm{W}_3(q)$.*

Proof If $\delta < \varepsilon_q$, then any $\{\delta(q+1),\delta;3,q\}$-minihyper (F,w) can be written as a sum of lines, see [1]. Applying this result to the minihyper from the statement of Theorem 2.1, it follows that the set $\mathcal{H}^*$ can be extended to a hemisystem of $\mathrm{W}_3(q)$ by adding the lines that constitute the sum.

$\boxed{\text{PROBLEM}}$ What if one or more of these lines that should be added are already lines of $\mathcal{H}^*$? Is this possible? Or if one or more of the lines that should be added are lines of $\mathrm{PG}(3,q)$ but not $\mathrm{W}_3(q)$? $\square$

Corollary 2.3 *If $\mathcal{H}$ is a partial hemisystem of $\mathrm{Q}(4,q)$ with deficiency $\delta < \varepsilon_q$, then $\mathcal{H}$ can be completed to a hemisystem of $\mathrm{Q}(4,q)$.*

Proof Remark that the second type of problematic lines in Corollary 2.2, the lines of $\mathrm{PG}(3,q)$ but not of $\mathrm{W}_3(q)$, do not pose a problem in this setting; they do not correspond to points of $\mathrm{Q}(4,q)$ and hence cannot extend the partial hemisystem. $\square$

Theorem 2.4 *Suppose $\mathcal{H}^*$ is a partial dual hemisystem of $\mathrm{H}(3,q^2)$. Define a weight function w as follows:*

$$w : \mathrm{PG}(3,q^2) \to \mathbb{N} : P \mapsto \begin{cases} 0 & \text{when } P \notin \mathrm{H}(3,q^2), \\ \frac{q+1}{2} - |\mathsf{star}(P) \cap \mathcal{H}^*| & \text{when } P \in \mathrm{H}(3,q^2). \end{cases}$$

If F is the set of points of $\mathrm{PG}(3,q^2)$ with positive weight, then (F,w) is a $\{\delta(q^2+1),\delta;3,q^2\}$-minihyper.

Proof The weight of $\mathrm{PG}(3,q)$ equals

$$w(\mathrm{PG}(3,q^2)) = \sum_{P\in \mathrm{PG}(3,q^2)} w(P) \;=\; \frac{q+1}{2}(q^2+1)(q^3+1) - |\mathcal{H}^*|(q^2+1)$$
$$= \delta(q^2+1).$$

A hyperplane π of $\mathrm{PG}(3,q)$ intersects $\mathrm{H}(3,q^2)$ either in a Hermitian curve $\mathrm{H}(2,q^2)$ or in cone over a variety $\mathrm{H}(1,q^2)$. In the former case, π contains no lines of $\mathrm{H}(3,q^2)$ and

$$w(\pi) = \sum_{P\in\pi} w(P) \;=\; \frac{q+1}{2}(q^3+1) - |\mathcal{H}^*|$$
$$= \delta.$$

In the latter case, π contains $q+1$ lines of $\mathrm{H}(3,q^2)$ that pass through a common point. Let α denote the number of lines of $\mathcal{H}^*$ contained in π. By definition, $\alpha \le (q+1)/2$. So,

$$w(\pi) = \sum_{P\in\pi} w(P) \;=\; \frac{q+1}{2}(q^3+q^2+1) - \alpha(q^2+1) - (|\mathcal{H}^*| - \alpha)$$
$$= \delta + q^2\left(\frac{q+1}{2} - \alpha\right) \;\ge\; \delta.$$

In both cases, $w(\pi)$ is at least δ.

A theorem of Hamada [2] shows that (F,w) is a $\{\delta(q^2+1),\delta;3,q^2\}$-minihyper. $\square$

Corollary 2.5 *If $\mathcal{H}^*$ is a partial dual hemisystem of $\mathrm{H}(3,q^2)$ with deficiency $\delta < \varepsilon_q$, then $\mathcal{H}^*$ can be completed to a dual hemisystem of $\mathrm{H}(3,q^2)$.*

Proof See the proof of Corollary 2.2.

$\boxed{\text{PROBLEM}}$ Note that the lines to be added are necessarily lines of $\mathrm{H}(3,q^2)$. But what if one or more of these lines that should be added are already lines of $\mathcal{H}^*$? Is this possible? $\square$

Corollary 2.6 *If $\mathcal{H}$ is a partial hemisystem of $\mathrm{Q}^-(5,q)$ with deficiency $\delta < \varepsilon_q$, then $\mathcal{H}$ can be completed to a hemisystem of $\mathrm{Q}^-(3,q)$.*

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