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No-arbitrage criteria for financial markets with efficient friction

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Abstract. We consider a multi-asset discrete-time model of a financial market with proportional transaction costs and efficient friction and prove necessary and sufficient conditions for the absence of arbitrage. Our main result is an extension of the Dalang–Morton–Willinger theorem. As an application, we establish a hedging theorem giving a description of the set of initial endowments which allows to super-replicate a given contingent claim.

Key words: Transaction costs, arbitrage, hedging, solvency

JEL Classification: G13, G11

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1 Introduction

The famous result, sometimes referred to as the First Fundamental Theorem on Asset (or Arbitrage) Pricing (FTAP) asserts that a frictionless financial market is arbitrage-free if and only if the price process is a martingale under a probability measure equivalent to the objective one. The original formulation due to Harrison and Pliska [6] involves the assumption that the underlying probability space $(\Omega, \mathscr{F}, P)$ (in other words, the number of states of the nature) is finite; it has been removed in the subsequent study of Dalang et al. [2]. Surprisingly,

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the passage from finite to infinite Ω is not a simple exercise: instead of purely geometric considerations (which make the Harrison–Pliska theorem so attractive for elementary courses in financial economics) much more delicate topological or measure-theoretical arguments must be used. These mathematical aspects attracted attention of a number of authors and new nontrivial equivalences were added (see, e.g., [22], [19], [12], [18], [7]). Now the no-arbitrage criteria in the absence of friction are well-understood and simple proofs are available, [15], as well as deep extensions to the continuous-time setting, [4], [5], [9]. The aim of this paper is to present no-arbitrage criteria for a multi-asset multi-period model with proportional transaction costs complementing the results of the note [14] where the case of finite Ω was treated and theorems, reducing to the classical Harrison–Pliska theorem were established, see updated versions in [11]. We use the geometric formalism developed in [10], [3], and [13]. In these papers it was shown that the concept of equivalent martingale measures, so useful in the context of frictionless market models, has to be changed for a concept of "dual" variables, which are, in the case of frictionless market, unnormalized martingale densities.

Slightly abusing the terminology of [15], we may formulate our main conclusion, Theorem 1, as follows:

In the presence of efficient friction, a financial market does not admit weak arbitrage opportunities at any date if and only if there exists a dual martingale process evolving in the interior of the positive dual to the solvency cone.

Although the literature on models with transaction costs is rapidly growing, there are only a few papers dealing with necessary and sufficient conditions for the absence of arbitrage. The article [8] contains an interesting approach which is different from ours not only at the level of modeling (continuous-time setting with the bid and ask prices) but also in the formulation of the no-arbitrage criteria. An attempt to find an arbitrage pricing theorem (for the binomial model) can be found in the preprint [20].

Addressing here the readers who are interested also in mathematical structures, we adopt, in contrast to [14], an abstract formulation, which makes clear that the basic model is a particular case of a linear regulator with random coefficients and specific conic constraints. No-arbitrage conditions can be formulated as certain properties of the attainability set of the corresponding linear system.

We end this paper by a section devoted to hedging theorems giving "dual" descriptions of the initial wealth which allows the investor to hedge successfully contingent claims without any risk, just by super-replication. Mathematically, the key issue here is the closedness of the set of subgains and this is one of the reasons why no-arbitrage criteria are considered as important results. The principal result of the paper implies an improvement of the hedging theorem [3]: the existence of the equivalent martingale measure, i.e. the no-arbitrage condition without friction is replaced by a certain no-arbitrage property involving transaction costs. Unfortunately, we pay for this progress: at the moment, we can guarantee the sufficiency of this property only assuming the efficient friction (therefore, our theorem does not imply that of Dalang–Morton–Willinger).

Remarks on notations. We shall work in a framework where $(\Omega, \mathscr{F}, P)$ is a complete probability space equipped with a finite discrete-time filtration $\mathbf{F} = (\mathscr{F}_t)$, $t = 0, 1, \ldots, T$; the σ-algebras $\mathscr{F}_t$ are assumed to be completed. For a process $X = (X_t)$ we define $X_- := (X_{t-1})$, $\Delta X_t := X_t - X_{t-1}$ with suitable conventions for X_{-1}. Clearly, every process X can be restored by its initial value and the increment process.

If $\omega \mapsto N(\omega) \subseteq \mathbf{R}^d$ is a set-valued mapping, then $L^0(N, \mathscr{F}_t)$ will denote the set of all $\mathscr{F}_t$-measurable selectors of N (we shall omit $\mathscr{F}_T$ in notations of this kind). In particular, $L^0(\mathbf{R}^d)$ is the space of random vectors. If A is a set of random variables then $\overline{A}$ is the closure with respect to the convergence in probability (or a.s.).

2 Portfolio processes under friction

We consider a financial market with d traded securities (e.g., currencies) with the prices given by an $\mathbf{R}^d$-valued adapted process S with strictly positive components. As in [14], we do not assume that the reference asset is a traded security. By convention, $S_{-1} = S_0$.

The agent's positions at time t can be described either by a vector $\widehat{V}_t$ of "physical units", or by a vector V_t of values invested in each position. These two vectors are related in the obvious way: $V_t^i = \widehat{V}_t^i S_t^i$, $i = 1, \ldots, d$.

The market friction is given by an adapted process Λ (of transaction costs coefficients) with values in the set $\mathbf{M}_+^d$ of matrices with non-negative entries and zero diagonal.

A "comprehensive" description of the portfolio dynamics (in values) can be done in terms of the increments as follows:

$$\Delta V_t^i = \widehat{V}_{t-1}^i \Delta S_t^i + \Delta B_t^i, \quad i = 1, \ldots, d, \; t = 0, 1, \ldots, T, \tag{1}$$

where $V_{-1}^i = v^i$,

$$\Delta B_t^i := \sum_{j=1}^d \Delta L_t^{ji} - \sum_{j=1}^d (1 + \lambda_t^{ij}) \Delta L_t^{ij}, \tag{2}$$

$\Delta L_t^{ji} \geq 0$ is an $\mathscr{F}_t$-measurable random variable representing the net value transferred to the position i from the position j. In other words, the increment ΔV_t^i of the value invested into the i-th position consists of two parts: the increment $\widehat{V}_{t-1}^i \Delta S_t^i$ due to the price changes and the increment ΔB_t^i due to the agent's action at time t. A consecutive choice of matrices ΔL_t with non-negative entries (depending on the history up to the time t) forms the agent's strategy.

The dimension of the action space can be radically reduced. To this end, we rewrite the portfolio dynamics in a way which makes clear that it is given by a very simple linear controlled finite difference equation with controls satisfying conic constraints. For each ω, t we consider in $\mathbf{R}^d$ the set

$$M_t(\omega) := \left\{ x : \ \exists\, a \in \mathbf{M}^d_+ \text{ such that } x^i = \sum_{j=1}^{d} [(1 + \lambda^{ij}_t(\omega))a^{ij} - a^{ji}],\ i \le d \right\}$$

which is a polyhedral cone as the image of the polyhedral cone $\mathbf{M}^d_+$ under a linear mapping; it has, at most, $d \times (d-1)$ generators. Its dual positive cone has the following representation:

$$M^*_t(\omega) = \{ w \in \mathbf{R}^d : \ w^j - (1 + \lambda^{ij}_t(\omega))w^i \le 0,\ 1 \le i,j \le d \}.$$

Introducing the process Y with

$$\Delta Y^i_t = \frac{1}{S^i_{t-1}} \Delta S^i_t, \quad Y^i_0 = 1,$$

we represent the portfolio dynamics as a linear controlled system

$$\Delta V^i_t = V^i_{t-1} \Delta Y^i_t + \Delta B^i_t, \quad i = 1, \dots, d, \tag{3}$$

where B belongs to $\mathscr{B}$, the set of processes with $\Delta B_t \in L^0(-M_t, \mathscr{F}_t)$, $t = 0, \dots, T$ (by convention, $B_{-1} = 0$). We leave to the reader to check (using measurable selection arguments) that this system generates the same set of value processes as given by (1), i.e. to verify that any $\Delta B_t \in L^0(-M_t, \mathscr{F}_t)$ can be obtained via (2) with some $\Delta L_t \in L^0(\mathbf{M}^d_+, \mathscr{F}_t)$.

We denote by $\mathscr{V}^0$ the set of all process $V = V^B$ with initial value ($V_{-1} = 0$) and increments given by (3) where B runs through $\mathscr{B}$. Put $R_t := \{V_t : \ V \in \mathscr{V}^0\}$. The set R_t describes the "results" or "gains" which can be obtained at the date t starting from the zero initial endowment. We introduce also the set

$$A_t := R_t - L^0(\mathbf{R}^d_+, \mathscr{F}_t)$$

of "hedgeable claims" or "subgains".

It is useful to make a look at the corresponding objects in terms of "physical" units of assets.

Define the diagonal operators

$$\phi_t(\omega) : (x^1, \dots, x^d) \mapsto (x^1/S^1_t(\omega), \dots, x^d/S^d_t(\omega))$$

preserving the cone $\mathbf{R}^d_+$. We may write that $\widehat{V}_t = \phi_t V_t$ and use in the sequel the abbreviations $\widehat{M}_t = \phi_t M_t$, $\widehat{A}_t = \phi_t A_t$, etc.

Notice that $\Delta \widehat{V}_t = \widehat{\Delta B_t}$ (this formula is obvious from the financial point of view but its formal check is also simple). That is $\widehat{V} = \tilde{B}$ where

$$\Delta \tilde{B}_t = \widehat{\Delta B_t} \in L^0(-\widehat{M}_t, \mathscr{F}_t).$$

Thus, we have a bijection between the sets of processes V and $\widehat{V}$.

Surely, the evolution of the process $\widehat{V}$ is much simpler: it is given by a relation which is not an equation. Of course, there is nothing new here: the expression $V^i_t = S^i_t \widehat{V}^i_t$ is just the well-known formula for the solution of non-homogeneous linear difference equation (written for (3) in this strange cryptic form).

As it was shown in [14] and [11], for models with market friction the concept of arbitrage admits various natural generalizations. The solvency region plays here an important role.

Let $K_t := \mathbf{R}^d_+ + M_t$ and $F_t := K_t \cap (-K_t)$. The sets $K_t(\omega)$ are polyhedral cones (hence closed), $F_t(\omega)$ are linear spaces. Clearly, the cone $K_t(\omega)$ is the *solvency region* (in values), being formed by vectors which can be transformed in a vector with only non-negative components by a certain transform, i.e. by adding a vector from $-M_t(\omega)$, while $F_t(\omega)$ represents positions which can be converted into zero and vice versa (necessarily, these two transactions are free of charge).

We shall say that a strategy B is a *weak arbitrage* opportunity at time t if $V_t^B \in K_t$ a.s. but $P(V_t \in K_t \setminus F_t) > 0$.

The absence of weak arbitrage opportunities (i.e. *strict no-arbitrage property*) at date t can be expressed in geometric terms:

NA$_t^s$. $R_t \cap L^0(K_t, \mathscr{F}_t) \subseteq L^0(F_t, \mathscr{F}_t)$.

It is an easy exercise to check that the above inclusion can be replaced by the following equivalent one:

$$A_t \cap L^0(K_t, \mathscr{F}_t) \subseteq L^0(F_t, \mathscr{F}_t).$$

Without any difficulty one can formulate the **NA**$_t^s$ condition using the sets of gains (or “subgains”) and the solvency regions in “physical” units.

For the case of finite Ω the references [14] and [11] give criteria for **NA**$_T^s$ as well as for the weaker property

NA$_T^w$. $R_T \cap L^0(K_T, \mathscr{F}_T) \subseteq L^0(\partial K_T, \mathscr{F}_T)$.

These criteria coincide with the Harrison–Pliska theorem if $\Lambda = 0$. The question, whether the assumption that Ω is finite can be omitted in their formulations, remains open. In the present paper we report some progress, and provide, for an arbitrary Ω, necessary and sufficient conditions of the absence of weak arbitrage opportunities along the whole time interval (**NA**$_T^s$ does not imply **NA**$_t^s$ for $t < T$, see an example in [14]). We shall work assuming the condition of efficient friction formulated as follows:

EF. The cones $K_t(\omega)$ are proper, i.e. $F_t(\omega) = \{0\}$ for every (ω, t).

Equivalently, one can say that every $K_t^*(\omega)$ has the non-empty interior.

The economic interpretation of weak arbitrage opportunities is obvious for the case where all $\lambda^{ij} > 0$, $i \neq j$ (and, hence, **EF** holds). If someone has an access to a market with smaller transaction costs, such an agent can transform a nonzero position in K to a positive gain.

Under **EF** the property of interest is

NA$^{\mathbf{s}}$. $R_t \cap L^0(K_t, \mathscr{F}_t) = \{0\}$ for $t = 0, 1, \ldots, T$.

It is easy to check (e.g., by examining K_t^* and M_t^*) that the condition **EF** implies that $K_t = M_t$.

Our main result is

Theorem 1 *Assume that* **EF** *holds. Then the following conditions are equivalent:*

(a) $\mathbf{NA^s}$*;*
(b) $A_t \cap L^0(K_t, \mathscr{F}_t) = \{0\}$ *for all* $t \le T$*;*
(c) $\underline{A}_t \cap L^0(K_t, \mathscr{F}_t) = \{0\}$ *and* $A_t = \underline{A}_t$ *for all* $t \le T$*;*
(d) $\bar{A}_t \cap L^0(K_t, \mathscr{F}_t) = \{0\}$ *for all* $t \le T$*;*
(e) *there exists a bounded martingale* Z *such that* $Z_s \in L^0(\operatorname{int}(\widehat{K}_s)^*, \mathscr{F}_s)$, $s \le T$*;*
(f) *for every* $t \le T$ *there exists a bounded martingale* $Z^t = (Z_s^t)_{s \le t}$ *such that* $Z_s^t \in L^0((\widehat{K}_s)^*, \mathscr{F}_s)$, $s \le t$, *and* $Z_t^t \in L^0(\operatorname{int}(\widehat{K}_t)^*, \mathscr{F}_t)$.

Proof Without loss of generality we may assume that S (hence Y) is identically equal to $\mathbf{1} := (1, \dots, 1)$. To see this, notice that one can replace R_t by $\widehat{R}_t = \phi_t R_t$ in the condition (a) of the theorem. On the other hand, the sets $\widehat{R}_t$ are generated by the controlled processes $\widehat{V}$ with dynamics given by (3) where Y is constant.

But for the case $Y = 1$ the claim follows easily from the Theorem 3 on separation of random polyhedral cones given in the next section. □

Notice that the components of martingales in (e) and (f) are strictly positive.

Remark The reader can easily add to this list a number of reformulations (e.g., expressing $\mathbf{NA^s}$ in terms of units of assets).

3 Sums of closed convex cones

We start with the following simple observation. Let $K = N_1 + N_2$ where N_i are closed convex cones in $\mathbf{R}^d$. If the cone K is proper, i.e. $K \cap (-K) = \{0\}$, or if only $N_1 \cap (-N_2) = \{0\}$, then K is closed. Indeed, let $x_1^n + x_2^n \to x$ where $x_i^n \in N_i$. If $\liminf |x_1^n| < \infty$ then there is a subsequence n_k such that $x_1^{n_k}$ converge to some $x_1 \in N_1$. Also $x_2^{n_k}$ converge to some $x_2 \in N_2$. The relation $x = x_1 + x_2$ shows that $x \in K$. The case $\liminf |x_1^n| = \infty$ is impossible: the sequence $\tilde{x}_1^n := x_1^n/|x_1^n|$ contains a subsequence $x_1^{n_k}$ converging to a certain $\tilde{x}_1 \in N_1$ with $|\tilde{x}_1| = 1$. But $\tilde{x}_1 = -\tilde{x}_2$, where $\tilde{x}_2 := \lim x_2^{n_k}/|x_1^{n_k}|$ is in N_2, contradicting the assumption.

Combining these arguments with Lemma 1 below we establish Theorem 2 on closedness of sums of convex cones in L^0 playing an important role in the proof of no-arbitrage criteria.

Lemma 1 *Let* $\eta^n \in L^0(\mathbf{R}^d)$ *be a sequence with* $\eta_* := \liminf |\eta^n| < \infty$. *Then there is an increasing sequence of* $\mathbf{N}$*-valued random variables* τ_n *such that the sequence* η^{τ_n} *converges (for almost all* ω*).*

Proof Let $\sigma(0) := 0$ and $\sigma(k) := \inf\{n > \sigma(k-1) : \; ||\eta^n| - \eta_*| \le 1/k\}$. For the sequence $\tilde{\eta}^n := \eta^{\sigma(n)}$ we will have $\sup_n |\tilde{\eta}^n| < \infty$. In particular, $\eta_*^1 := \liminf \eta^{1n} < \infty$. Let

$$\tau_1(0) := 0, \quad \tau_1(k) := \inf\{n > \tau_1(k-1) : \; |\tilde{\eta}^{1n} - \eta_*^1| \le 1/k\}, \quad k \ge 1.$$

In a similar way, working with the second component of the sequence $\tilde{\eta}^{\tau_1(n)}$ whose first component is convergent, we construct an increasing sequence $\tau_2(k)$ and so on. Obviously, the sequence $\tau_n := \sigma \circ \tau_1 \circ \dots \circ \tau_d(n)$ has the claimed property. □

Lemma 2 *Assume that N is a closed convex cone in L^0 stable under multiplication by the nonnegative random variables. If τ is an $\mathbf{N}$-valued random variable and $\xi^n \in N$, $n \in \mathbf{N}$, then $\xi^\tau \in N$.*

Proof The stability under multiplication ensures that $\xi^n I_{\{\tau=n\}} \in N$. Since N is a closed cone, it contains $\xi^\tau = \sum \xi^n I_{\{\tau=n\}}$. □

Let $A_t := \sum_{s=0}^{t} N_s$ where N_s are subsets of a vector space E.

Lemma 3 *Let N_s be convex cones. Introduce the following conditions:*

(i) $A_T \cap (-N_t) = \{0\}$ *for every* $t = 0, ..., T$;
(ii) $A_{t-1} \cap (-N_t) = \{0\}$ *for every* $t = 1, ..., T$;
(iii) the relation $\sum_{s=0}^{T} x_s = 0$ *with* $x_s \in N_s$ *holds iff all* $x_s = 0$.

Then $(i) \Rightarrow (ii) \Leftrightarrow (iii)$. *If the cones N_s are proper, all conditions are equivalent.*

Proof

(i) $\Rightarrow (ii)$ Trivial because $A_{t-1} \subseteq A_T$.
(ii) $\Rightarrow (iii)$ Assume that $\sum_{s=0}^{T} x_s = 0$ with $x_s \in N_s$ not all equal to zero. Let t be the largest index for which $x_t \neq 0$. The relation $\sum_{s=0}^{t-1} x_s = -x_t$ contradicts to $A_{t-1} \cap (-N_t) = \{0\}$.
(iii) $\Rightarrow (ii)$ Obvious.

The implication $(iii) \Rightarrow (i)$ when all N_s are proper cones is also easy. If (i) does not hold, then $\sum_{s=1}^{T} y_s = -z_t$ where $z_t \neq 0$ is an element of some N_t and $y_s \in N_s$ are not all equal to zero. If there is $y_s \neq 0$, $s \neq t$, the contradiction with (iii) is clear. But the case where only $y_t \neq 0$ is impossible because the cone N_t is proper. □

Theorem 2 *Let N_s be closed convex cones in $L^0(\mathbf{R}^d, \mathcal{F}_s)$ stable under multiplication by the elements of $L^0(\mathbf{R}_+, \mathcal{F}_s)$. If $A_{t-1} \cap (-N_t) = \{0\}$ for every $t = 1, ..., T$, then $A_T = \overline{A}_T$ and hence $A_T \cap L^1(\tilde{P})$ is closed in $L^1(\tilde{P})$ for every $\tilde{P} \sim P$.*

Proof We proceed by induction. Assume that the assertion holds for $T-1$. In particular, the set $\sum_{s=1}^{T} N_s$ is closed. Let $\sum_{s=0}^{T} \xi_s^n \to \xi$ a.s. where $\xi_s^n \in N_s$. Introduce the set $\Gamma := \{\liminf |\xi_0^n| < \infty\}$ and define the random variables $\xi_s'^n = \xi_s^n I_\Gamma$ which are in N_s. By Lemma 1 there is an increasing sequence of $\mathbf{N}$-valued $\mathcal{F}_0$-measurable random variables τ_n such that $\xi_0'^{\tau_n}$ converge to $\xi_0' \in N_0$. Then $\sum_{s=1}^{T} \xi_s'^{\tau_n}$ converge a.s. to a random variable $\zeta \in \sum_{s=1}^{T} N_s$ vanishing on Γ^c and such that $\xi I_\Gamma = \xi_0' + \zeta$.

It remains to prove that $\xi I_\Gamma = \xi$ a.s. Put $\tilde{\xi}_s^n := (\xi_s^n / |\xi_0^n|) I_{\Gamma^c}$ (using the convention $0/0 = 0$). Since $|\tilde{\xi}_0^n| \leq 1$, there is an increasing sequence of $\mathbf{N}$-valued $\mathcal{F}_0$-measurable random variables σ_n such that $\tilde{\xi}_0^{\sigma_n}$ converge to some $\tilde{\xi}_0 \in N_0$. Then $\sum_{s=1}^{T} \tilde{\xi}_s^{\sigma_n}$ converge a.s. to a random variable $\tilde{\zeta}$. By the induction hypothesis $\tilde{\zeta} = \sum_{s=1}^{T} \tilde{\xi}_s$ where $\tilde{\xi}_s \in N_s$. Notice that $\xi / |\xi_0^n| \to 0$ a.s. on Γ^c. Thus, $\sum_{s=0}^{T} \tilde{\xi}_s = 0$ where $\tilde{\xi}_s \in N_s$. By Lemma 3 all $\tilde{\xi}_s = 0$. Since $|\tilde{\xi}_0| = 1$ on Γ^c, we conclude that $P(\Gamma^c) = 0$. □

The following useful assertion is almost obvious:

Lemma 4 *Let Z be an $\mathbf{R}^d$-valued martingale and let $\Sigma_T := Z_T \sum_{s=0}^T \xi_s$ where $\xi_s \in L^0(\mathbf{R}^d, \mathscr{F}_s)$ are such that $Z_s \xi_s \le 0$. If the negative part of Σ_T is integrable then $E\Sigma_T \le 0$.*

Proof. For $T = 0$ this is obvious. Assume that the claim is true for $T-1$. Clearly,

$$Z_T \sum_{s=0}^{T-1} \xi_s \ge -\Sigma_T^- - Z_T \xi_T \ge -\Sigma_T^- .$$

By conditioning we get that

$$Z_{T-1} \sum_{s=0}^{T-1} \xi_s \ge -E(\Sigma_T^- | \mathscr{F}_{T-1}).$$

Hence, by the induction hypothesis $E\Sigma_{T-1} \le 0$. As $Z_T \xi_T \le 0$, we get the result. □

Lemma 5 *Let N_s be subsets of $L^0(\mathbf{R}^d, \mathscr{F}_s)$. Suppose that for each $t \le T$ there exists a $\mathbf{R}^d$-valued martingale Z^t with the following properties:*

1) $Z_s^t \xi \le 0$ for every $\xi \in N_s$, $s \le t$;
2) the equality $Z_t^t \xi = 0$ where $\xi \in N_t$ holds iff $\xi = 0$.

Then $A_{t-1} \cap (-N_t) = \{0\}$ for every $t = 1, \dots, T$.

Proof If the assertion fails to be true, there are $\xi_s \in N_s$, $s \le t$, such that $\xi_t \ne 0$ and $\sum_{s=0}^{t-1} \xi_s = -\xi_t$. Then

$$Z_t^t \sum_{s=0}^{t-1} \xi_s = -Z_t^t \xi_t \ge 0.$$

By the above lemma

$$EZ_t^t \sum_{s=0}^{t-1} \xi_s \le 0.$$

It follows that $Z_t^t \xi_t = 0$. In virtue of 2) this is possible only if $\xi_t = 0$. □

Lemma 6 *Let N_s be closed convex cones in $L^0(\mathbf{R}^d, \mathscr{F}_s)$ stable under multiplication by the elements of $L^0(\mathbf{R}_+, \mathscr{F}_s)$. Assume that $A_T \cap (-N_t) = \{0\}$ for every $t = 0, \dots, T$. Then for any $\zeta \in N_t$, $t \le T$, there is a bounded $\mathbf{R}^d$-valued martingale Z^ζ such that:*

1) $Z_s^\zeta \xi \le 0$ for any $\xi \in N_s$, $s \le T$;
2) $\{Z_t^\zeta \zeta < 0\} = \{\zeta \ne 0\}$ (a.s.).

Proof Put $A_T^1 := A_T \cap L^1$, $\mathscr{Z}_T := \{\eta \in L^\infty(\mathbf{R}^d, \mathscr{F}_T) : \ E\eta\xi \le 0, \ \xi \in A_T^1\}$. With any $\eta \in \mathscr{Z}_T$ we associate the martingale $Z_s := E(\eta|\mathscr{F}_s)$. It satisfies 1): otherwise we would find $\xi \in N_s \cap L^1$ such that the set $\Gamma := \{Z_s\xi > 0\}$ is of positive probability and hence $E\eta(\xi I_\Gamma) = EZ_s(\xi I_\Gamma) > 0$ contradicting the definition of $\mathscr{Z}_T$. Let $a := \sup_{\eta\in\mathscr{Z}_T} P(Z_t\zeta < 0) > 0$ (for a=0 there is nothing to prove). There is $\eta^* = \eta^*(\zeta) \in \mathscr{Z}_T$ such that for the corresponding martingale Z^ζ we have $a = P(Z_t^\zeta\zeta < 0)$. To see this, take $\eta_n \in \mathscr{Z}_T$ with $||\eta_n||_\infty = 1$ such that $P(Z_t^n\zeta < 0) \to a$ and put $\eta^* := \sum 2^{-n}\eta_n$.

Assume that the set $D := \{\zeta \ne 0\}\setminus\{Z_t^\zeta\zeta < 0\}$ is of positive probability. For c sufficiently large, the element $-I_{D\cap\{|\zeta|\le c\}}\zeta$ is nonzero and, being in $(-N_t)\cap L^1$, it is not in the convex cone A_T^1 which is closed in L^1 in virtue of Theorem 2. By the Hahn–Banach separation theorem there exists $\eta \in L^\infty(\mathbf{R}^d)$ such that

$$E\eta\xi < -E\eta I_{D\cap\{|\zeta|\le c\}}\zeta \quad \forall\, \xi \in A_T^1.$$

It follows that $E\eta\xi \le 0 \ \forall\, \xi \in A_T^1$ (i.e. $\eta \in \mathscr{Z}_T$) and $E\eta I_{D\cap\{|\zeta|\le c\}} < 0$. Thus, for $\tilde{Z}$ corresponding to $\tilde{\eta} := \eta^* + \eta$ we have

$$P(\tilde{Z}_t\zeta < 0) > P(Z_t^\zeta\zeta < 0) = a.$$

This contradiction shows that 2) holds. □

Corollary 1 *In the setting of Lemma 6 for any countable set $\Gamma \subseteq \cup_{s\le T} N_s$ there is a bounded $\mathbf{R}^d$-valued martingale Z such that*

1) $Z_s\xi \le 0$ for any $\xi \in N_s$, $s \le T$;
2) $\{Z_t\zeta < 0\} = \{\zeta \ne 0\}$ (a.s.) when $\zeta \in \Gamma \cap N_t$.

The proof is standard: one can choose Z as an appropriate countable convex combination of Z^ζ, $\zeta \in \Gamma$.

We say that a sequence of set-valued mappings $K = (K_t)$ is a $\mathscr{C}$-valued process if there is a countable sequence of adapted $\mathbf{R}^d$-valued processes $X^n = (X_t^n)$ such that for every t and ω only a finite but non-zero number of $X_t^n(\omega)$ is different from zero and $K_t(\omega) := \text{cone}\,\{X_t^n(\omega),\ n \in \mathbf{N}\}$ (i.e. $K_t(\omega)$ is a polyhedral cone generated by the finite set $\{X_t^n(\omega),\ n \in \mathbf{N}\}$).

Remark Using standard facts on measurable selection (see the book [1] for references) one can show that K is a $\mathscr{C}$-valued process if and only if each K_t is $\mathscr{F}_t$-measurable mapping taking values in the set of polyhedral cones in $\mathbf{R}^d$.

Obviously, the sequence of solvency cones $K = (K_t)$ in our model of financial market with friction is a $\mathscr{C}$-valued process.

Now we summarize equivalent properties in the specific setting which is of our primary interest.

Theorem 3 *Let G be a $\mathscr{C}$-valued process such that all cones $G_s(\omega)$ are proper. Let $A_t := \sum_{s=0}^t N_s$ where $N_s := -L^0(G_s, \mathscr{F}_s)$. Then the following conditions are equivalent:*

(a) $A_t \cap (-N_t) = \{0\}$, $t = 0, \dots, T$;
(b) *there exists a martingale* Z *such that* $Z_s \in L^\infty(\operatorname{int} G_s^*, \mathscr{F}_s)$ *for each* $s \le T$;
(c) *for each* $t \le T$ *there is a martingale* Z^t *such that* $Z_s^t \in L^\infty(G_s^*, \mathscr{F}_s)$ *for each* $s \le t$ *and* $Z_t^t \in L^\infty(\operatorname{int} G_s^*, \mathscr{F}_s)$

Proof The implication $(b) \Rightarrow (c)$ is trivial. The property (a) (equivalent to the properties in the formulation of Lemma 3) ensures (Theorem 2) that $A_T = \bar{A}_T$. Notice that in our setting for each t there is a family $\zeta_t^i \in -N_t$, $i \in \mathbf{N}$, such that the set $\{\zeta_t^i(\omega),\ i \in \mathbf{N}\}$ is finite and generates the cone $G_t(\omega)$ (a.s.). For the union Γ of these families we can find a bounded martingale Z satisfying the conditions of Corollary 1. This proves that $(a) \Rightarrow (b)$. The implication $(c) \Rightarrow (a)$ follows from Lemmas 5 and 3. □

4 Hedging theorems

At first, we establish an "abstract" version of the hedging theorem giving a description of the set of initial endowments starting from which the investor can cover the future pay-off by the terminal wealth of a value process.

Omitting ω, we shall denote by "$\ge_T$" the partial ordering generated by G_T (i.e. $x \ge_T y \Leftrightarrow x - y \in G_T$).

Let ϑ be an $\mathbf{R}^d$-valued random variable (interpreted as a contingent claim in units) such that $\vartheta \ge_T -c\mathbf{1}$ for some constant c.

We consider the setting of the previous theorem and assume moreover that the initial σ-algebra is trivial.

Let $\mathscr{Z}$ (resp., $\mathscr{Z}^0$) be the set of martingales Z such that $Z_s \in L^\infty(G_s^*, \mathscr{F}_s)$ (resp., $Z_s \in L^\infty(\operatorname{int} G_s^*, \mathscr{F}_s)$) for each $s \le T$.

Let us consider the following convex sets in $\mathbf{R}^d$:

$$
\begin{aligned}
H_1 &:= \{v \in \mathbf{R}^d : \ \vartheta \in v + A_T\}, \\
H_2 &:= \{v : \ EZ_T\vartheta \le Z_0 v, \ \forall Z \in \mathscr{Z}^0\}.
\end{aligned}
$$

Clearly, H_2 is always closed while H_1 is closed simultaneously with A_T. As we have shown, the latter is closed if there is $Z^o \in \mathscr{Z}^0$. Arguing with bounded martingales of the form $(1-\varepsilon)Z + \varepsilon Z^o$ and letting ε decrease to zero, we conclude that $\mathscr{Z}^0$ in the definition of H_2 can be replaced by $\mathscr{Z}$ if $\mathscr{Z}^0 \ne \emptyset$.

Theorem 4 *Assume that* $\mathscr{Z}^0 \ne \emptyset$. *Then* $H_1 = H_2$.

Proof Let $v \in H_1$. This means that

$$
\vartheta = v + \sum_{s=0}^{T} \xi_s, \qquad \xi_s \in -L^0(G_s, \mathscr{F}_s).
$$

The inclusion $H_1 \subseteq H_2$ follows from Lemma 4 because for $Z \in \mathscr{Z}^0$ we have

$$
Z_T \sum_{s=0}^{T} \xi_s \ge Z_T\vartheta - Z_T v \ge -c|Z_T| - Z_T v.
$$

and the right-hand side of the last inequality is an integrable random variable.

Suppose that the converse inclusion $H_2 \subseteq H_1$ fails. This means that there is $v \in H_2$ such that the point $\vartheta - v$ does not belong to the convex closed set A_T. We choose a probability measure $\tilde{P} \sim P$ with the bounded density $f := d\tilde{P}/dP$ such that ϑ is in $L^1(\mathbf{R}^d, \tilde{P})$. Then $\vartheta - v$ can be separated from $A_T \cap L^1(\mathbf{R}^d, \tilde{P})$, i.e. there exists $\eta \in L^\infty(\mathbf{R}^d)$ such that

$$\tilde{E}\eta(\vartheta - v) > \tilde{E}\eta\zeta, \quad \forall \zeta \in A_T \cap L^1(\mathbf{R}^d, \tilde{P}).$$

It follows that $\tilde{E}\eta(\vartheta - v) > 0$ while $\tilde{E}\eta\zeta \leq 0$ for all $\zeta \in A_T \cap L^1(\mathbf{R}^d, \tilde{P})$. The standard arguments show that $Z_t = E(f\eta|\mathscr{F}_t)$ is an element of $\mathscr{Z}$ and we arrive to a contradiction with the assumption $v \in H_2$. $\square$

The above theorem applied with $G_t = \widehat{K}_t$ gives directly the desired “dual” description of the set of initial endowments

$$H_1 = \{v \in \mathbf{R}^d \,:\, \exists\, V \in \mathscr{V}^{\cdot} \text{ such that } v + \widehat{V}_T \geq_T \vartheta\}$$

expressed in terms of units of traded currencies needed to hedge (i.e. super-replicate) a contingent ϑ also expressed in terms of units.

Now we explain the relation with the previously available result in [3] where the problem was formulated in terms of values, i.e. the components of a contingent claim C are liabilities in correspondent currencies measured in the reference asset. This contingent claim in terms of units will be $\vartheta = \phi_T C$.

The set of hedging endowments in values is defined in [3] as follows:

$$\Gamma := \{v \in \mathbf{R}^d \,:\, \exists\, V \in \mathscr{V}^{\cdot} \text{ such that } v + V_T \succeq_T C\}$$

where “$\succeq_T$” is the partial ordering generated by K_T. Clearly, $\Gamma = \phi_t^{-1}H_1$. Theorem 3 implies as a corollary

Theorem 5 *Suppose that the hypotheses* **EF** *and* $\mathbf{NA}^s$ *hold. Then*

$$\Gamma = D := \{v \in \mathbf{R}^d \,:\, E\widehat{Z}_T C \leq \widehat{Z}_0 v \;\forall Z \in \mathscr{Z}\}$$

where $\mathscr{Z}$ is the set of bounded martingales such that $\widehat{Z}_t \in L^0(K_t^*, \mathscr{F}_t)$ for $t = 0, \ldots, T$.

Notice that the formulation of Theorem 4.2 in [3] uses, instead of $\mathscr{Z}$ the set $\mathscr{D}$ of martingales Z for which $\widehat{Z}_T$ is bounded and $\widehat{Z}_t \in L^0(K_t^*, \mathscr{F}_t)$ for $t = 0, \ldots, T$. To get such a description one needs only a minor modification of the separation arguments.

Although the above theorem is established under the efficient friction condition, it has an advantage with respect to Theorem 4.2 in [3]: the **EMM** condition is replaced by the weaker (and seemingly more relevant) condition $\mathbf{NA}^s$. Notice also that $\mathbf{NA}^s$ is weaker than the hypothesis in the two-asset hedging theorem of [16].

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