Equivalent martingale measures for large financial markets in discrete time

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Abstract

We show that in a discrete-time large financial market the absence of certain asymptotic arbitrage opportunities is equivalent to the existence of martingale measures in a strong sense. We also consider the Arbitrage Pricing Model with stable random variables where we are able to give explicit necessary and sufficient conditions using market parameters.

Keywords: Large financial market; Asymptotic arbitrage; Martingale measures; APM; Stable random variables
Running title: Equivalent martingale measures

1 Introduction

The Arbitrage Pricing Model (APM) was first proposed by S. A. Ross. In [21] he considered a multifactor model with infinitely many assets. In the simplest one-factor case his main result asserts, roughly speaking, that the absence of certain asymptotic arbitrage opportunities implies that the asset returns are approximately linear functions of the correlation with the market portfolio; see [11, 13] for transparent expositions; see also section 3 of the present paper.

Investigations of [10, 1] shed light on the intimate relationship between no-arbitrage conditions and the existence of martingale measures in the framework of security markets in discrete time with finitely many assets. Later these questions were treated at increasing levels of generality in models with a continuous time parameter, see [24, 3, 4, 5, 12]. New "no free lunch" conditions have been introduced. All these developments relied on various versions and generalizations of the separation theorems of [26, 19], as does the present article.

Generalizing ideas of [21, 11], large financial markets were introduced in [14] as sequences of economies with an increasing (finite) number of assets. General results have been developed in a succession of papers: [17, 18, 15, 16].

In these articles various formulations of asymptotic arbitrage are considered. They are found to be equivalent to certain contiguity properties of sequences of martingale measures for the finite economies. As results are formulated for very general types of models, a restricted class of strategies is used: only those portfolios are admissible whose returns are (almost surely) uniformly bounded from below by a constant. Though in continuous-time models this restriction is a mathematical necessity (so as to rule out "doubling strategies"), it is not justified in discrete-time models.

The first main motivation of our research was the fact that in empirical studies asset returns are often modelled by random variables which are not bounded either from below or from above (see the models of section 3), and in this case the set of admissible strategies may reduce to the 0 portfolio. This means that the no-arbitrage criteria of the above mentioned papers are somewhat vacant when applied to such models since they are automatically satisfied. As an easy example, let us consider a market with a sequence of assets whose returns are independent Gaussian variables such that their expectation tends to infinity while their variance converges to 0. Intuitively, such a market contains enormous arbitrage opportunities, but none of the criteria of e.g. [15, 16] detects this, due to the fact that the chosen class of admissible strategies contains only 0 in the present case.

In an economy with finitely many assets equivalent martingale measures play a significant rôle: they give rise to positive linear pricing rules for contingent claims. Our second source of motivation was that we wondered under what kind of conditions we can obtain such measures for a discrete-time large financial market. In continuous-time markets this issue has been first treated by [16] for non-stationary models (i.e. where the finite asset economies may live on different probability spaces); [2] gave conditions for building martingale measures on stationary markets (i.e. where the finite asset economies are embedded in each other, see the definition in section 2) with a certain factor structure.

We shall be working in a market model with countably many assets and one time step, though everything carries over to multistep models, see Remark 2.2. The APM fits well into this framework. Trying to give a solution to the two problems explained above, we formulate a no-arbitrage criterion which uses the unrestricted set of portfolios and which is a direct generalization of the usual no-arbitrage condition for discrete-time markets with finitely many assets, see Remark 2.8. This criterion distinguishes e.g. Gaussian models as we shall illustrate in section 3. It will turn out to be equivalent to the existence of martingale measures in a certain strong sense, hence we get a (partial) answer to both of the questions explained above.

In section 2 we define the model we are working in, introduce the "no asymptotic free lunch" (NAFL) property and present our main result. Section 3 characterizes (NAFL) in the classical Arbitrage Pricing Model with stable random variables. Proofs appear in section 4. A few possible ramifications are discussed in the concluding section.

2 Absence of arbitrage and martingale measures

We now define a large financial market as a sequence of economies with an increasing number of assets. Our definition is only a very special case of the one in [14]. We fix a probability space $(\Omega, \mathcal{F}, P)$; all random variables are supposed to be defined on this space. The set of bounded random variables is denoted by L^∞.

Definition 2.1 *By large financial market we mean a sequence of "small markets" indexed by $k \in \mathbb{N}$, where prices of assets are observed at times 0 and 1. The price of the ith asset in the kth market at time $t \in \{0,1\}$ is a random variable*

$$S_k^i(t), \quad 0 \leq i \leq k.$$

We suppose that $S_k^i(0)$ is a constant. For the sake of simplicity we assume that each small market contains a riskless asset with constant price . Formally, we assume that there is a 0th asset and its price satisfies

$$S_k^0(t) := 1, \quad t \in \{0,1\}, \quad k \in \mathbb{N}.$$

We specify a portfolio strategy ϕ_k on the k-th market by a sequence

$$\phi_k^i, \ 1 \leq i \leq k$$

of real numbers which correspond to the investments in the respective assets at time 0. The position ϕ_k^0 in the 0th asset is now determined by the self-financing condition

$$\sum_{i=0}^{k} \phi_k^i = 0.$$

The return *on the portfolio* ϕ_k *is then*

$$V^{\phi_k} := \sum_{i=0}^{k} \phi_k^i (S_k^i(1) - S_k^i(0)) = \sum_{i=1}^{k} \phi_k^i (S_k^i(1) - S_k^i(0)).$$

Remark 2.2 *This setting may easily be generalized to multistep models with trading times* $0, \dots, T(k) \geq 1$. *In this case we assume that there are given filtrations* $(\mathcal{F}_t^k)_{0 \leq t \leq T(k)}$ *to which the process* S *is adapted. The set of portfolio strategies consists of stochastic processes* ϕ_k^i *where*

$$\phi_k^i(t), \quad 1 \leq i \leq k$$

are $\mathcal{F}_{t-1}$*-measurable,* $1 \leq t \leq T(k)$. *The return can now be defined as*

$$V^{\phi_k} := \sum_{t=1}^{T(k)} \sum_{i=1}^{k} \phi_k^i(t)(S_k^i(t) - S_k^i(t-1)).$$

We remark that in this more general situation our main result remains true with hardly any modification in the proof: one has to introduce C *as the set of returns on* bounded *portfolios and the proof of Theorem 2.7 goes through almost identically.*

We restrict ourselves to the one-step model so as to keep the presentation simple.

Our main result deals with a subclass of models in which the $(k+1)$th market is an enlargement of the kth, i.e. the small markets are embedded in each other. This motivates the following definition.

Definition 2.3 *We call the market* stationary *if*

$$S_{k+1}^i = S_k^i, \ 1 \leq i \leq k.$$

In this case we have a sequence of assets, so we can simplify the notation and write only

$$S_i, \ i \in \mathbb{N}.$$

During the proof we will work with the set of possible returns:

$$C := \{V^{\phi_k} : \ \phi_k \text{ is a portfolio in the } k\text{th market}, \ k \in \mathbb{N}\}. \tag{1}$$

Before continuing, we fix some notation: F denotes the set of $\mathbb{R}_+ \cup \{\infty\}$-valued random variables, L^∞ is the set of (almost surely) bounded random variables. Now we define our asymptotic no-arbitrage condition.

Definition 2.4 *A discrete-time market has* no asymptotic free lunch *(NAFL) if there exists no sequence of trading strategies* $(\phi_k)_{k\in\mathbb{N}}$ *such that*

$$V^{\phi_k} \xrightarrow{k\to\infty} V \text{ in probability}, \quad V \in F \setminus \{0\}.$$

In [16] the same term was used to denote a concept of absence of asymptotic arbitrage in a continuous-time semimartingale context. It had essentially the same content as the "no free lunch" (NFL) of e.g. [19]. As we deal with discrete-time models only we chose to call our condition in the same way as [16] despite the conceptual difference and referred to the one in [16] as (NFL).

The above concept is related to that of "no free lunch with bounded risk" (NFLBR), used in [3, 4, 23]. This latter condition states the non-existence of a sequence of returns uniformly bounded from below and converging in probability to some element of $F\setminus\{0\}$. The economic interpretation is straightforward: one can not have arbitrage in the limit with short-selling constraints. Compare also to (NAFLBR) in [16].

As indicated in the introduction, we wish to work with returns which are unbounded from below, hence the concept (NFLBR) had to be suitably adjusted. So let us investigate what (NAFL) means in economic terms. A sequence V^{ϕ_k} as in the above definition is not necessarily uniformly bounded from below. However, a subsequence (still denoted by k) converges almost surely to V, so the lower envelope

$$u := \wedge_{k\in\mathbb{N}} V^{\phi_k}$$

is almost surely finite. Agents populating the economy under consideration may have different beliefs and assessements of the market situation, corresponding to various subjective probabilities P'. We suppose, however, that they agree on zero-probability events, i.e. $P' \sim P$. An agent may have a subjective probability $P' \sim P$ which assigns very small weight to the set $\{u < -1\}$, hence for such a person pursuing the trading strategy $(\phi_k)_{k\in\mathbb{N}}$ may seem to be an appealing free lunch opportunity, as the possible loss is bounded by 1 with a probability very close to 1. This idea is related to the concept of quantile hedging, see [8]: in real market situations one often tries to super-replicate the desired claim with a certain probability only.

Roughly speaking, (NAFL) means that whatever the agents' assessments of the market situation might be (as long as they correspond to equivalent measures), one can not have any way of making something out of nothing with more or less "acceptable" risk.

Remark 2.5 *It is quite easy to check that (NAFL) implies the "no free lunch" (NFL) condition often encountered, e.g. in [19, 23, 12, 16]. The criteria (NFL) often furnishes martingale measures or related objects. Results of [16] imply that in a market with countably many assets such that each S_i is (locally) bounded (NFL) is equivalent to the existence of an equivalent martingale measure. There are certain drawbacks of (NFL) to be mentioned: it involves generalized sequences (nets), which are counter-intuitive and whose convergence seems to be difficult to verify in concrete models. These facts led to the creation of "sequential" asymptotic no-arbitrage conditions such as (NFLVR), (NFLBR), etc.; see [4, 15]. It is shown in [16], however, that these concepts are not strong enough to provide an equivalent martingale measure. (NAFL) is one possible strengthening of all these criteria, using (ordinary) sequences only.*

We now introduce two fundamental notions concerning equivalent martingale measures.

Definition 2.6 *If there is $Q \sim P$ such that*

$$E^Q S_i(1) = S_i(0), \quad i \in \mathbb{N}, \tag{2}$$

we say that there exists an equivalent martingale measure. We abbreviate this condition as (EMM). Indeed, (2) *states precisely that $(S_i(t))_{t\in\{0,1\}}$ is a Q-martingale, $i \in \mathbb{N}$. We say that* there exist equivalent martingale measures in the strong sense *(abbreviated (EMMSS)), if for all $P' \sim P$ there is $Q \sim P'$ with $dQ/dP' \in L^\infty$ and* (2) *holds.*

This can be interpreted in the following way: whatever the subjective probability $P' \sim P$ of the respective agent may be, there is a risk-neutral measure $Q \sim P'$ with bounded P'-density (such measures are sometimes called *uniform* martingale measures, see [7]).

Although (EMMSS) seems to be a rather technical condition, it is a straightforward generalization of the concept of equivalent martingale measure in the context of discrete-time markets with finitely many assets and finite time horizon; this will be pointed out in Remark 2.8. We have the following characterization, the main theorem of the paper.

Theorem 2.7 *In a stationary market (NAFL) is equivalent to (EMMSS).*

Remark 2.8 *The reader might have the impression that (NAFL) is too strong a requirement. Are there any non-trivial models satisfying it? From the above result we can immediately construct a simple example. Let us consider the stationary model with finitely many assets (say, $S_l = S_N,\ l \geq N$*

for a certain N). We recall from [1] the no-arbitrage *(NA) condition: if for any portfolio ϕ*

$$V^\phi \geq 0 \text{ a.s.} \Longrightarrow V^\phi = 0 \text{ a.s.},$$

then we say that (NA) holds. It is shown in [1] that (NA) is necessary and sufficient for the existence of an equivalent martingale measure. In fact, this measure can be chosen such that it has bounded P-density. As (NA) is not sensitive to an equivalent change of measure, we may conclude that in a model with finitely many assets (NA) is equivalent to (EMMSS), hence to (NAFL).

Example 2.9 *We give an easy example of an infinite market with (EMMSS). Let us suppose that in a stationary large financial market the increments $S_i(1) - S_i(0)$ are independent and symmetric. As convolutions of symmetric variables are themselves symmetric, we get that if for a sequence of portfolios ϕ_k we have*

$$V^{\phi_k} \xrightarrow{k\to\infty} V \in F,$$

in probability, then the negative parts $(V^{\phi_k})^-$ tend to 0 in distribution, hence the positive parts $(V^{\phi_k})^+$, too, and we obtain $V = 0$: the model enjoys the (NAFL) property.

Example 2.10 *It is possible that (EMM) holds while (EMMSS) fails. To see this, we take $\Omega = [0,1[$, $\mathcal{F} = \mathcal{B}([0,1[)$, P the Lebesgue measure. We define a stationary large financial market by*

$$S_i(0) := 1, \quad S_i(1) - S_i(0) := \Delta_i, \quad i \geq 1,$$

where for $k \geq 0$ and $2^k \leq i < 2^{k+1}$ we set

$$\Delta_i(t) := -1, \quad t \in \left[\frac{2i - 2^{k+1}}{2^{k+1}}, \frac{2i + 1 - 2^{k+1}}{2^{k+1}}\right[,$$
$$\Delta_i(t) := 1, \quad t \in \left[\frac{2i + 1 - 2^{k+1}}{2^{k+1}}, \frac{2i + 2 - 2^{k+1}}{2^{k+1}}\right[,$$
$$\Delta_i(t) = 0 \text{ otherwise.}$$

The reader can easily check that if Q is a martingale measure for all the market, i.e.

$$E^Q \Delta_i = 0, \; i \geq 1,$$

then the Q-measure of dyadic intervals of the form

$$I_l^k := \left[\frac{l}{2^k}, \frac{l+1}{2^k}\right[$$

is equal for $0 \le l < 2^k$, for each fixed $k \ge 0$. To put it another way, $Q(I_l^k) = P(I_l^k)$! A measure on $\mathcal{F}$ is determined by its values on dyadic intervals. As Q equals P on these we conclude that necessarily $Q = P$.

Now let us take any $P' \sim P$ with dP/dP' unbounded. Looking at the market under P', we conclude that (EMM) holds true (P is a martingale measure) but there is no martingale measure Q with $dQ/dP' \in L^\infty$. Thus we have found a model in which (EMM) holds while (EMMSS) fails. This example was suggested by Christophe Stricker.

3 APM with stable random variables

Now we present a class of infinite models in which we can characterize the (NAFL) property in terms of market parameters. We use a multifactor non-stationary version of the "revisited" APM in [15].

Definition 3.1 *We define for all $k \ge m$ the kth small market of the m-factor Arbitrage Pricing Model as*

$$S_k^0(0) = S_k^0(1) \equiv 1,$$
$$S_k^i(1) = S_k^i(0)(1 + \mu_k^i + \bar{\kappa}_k^i \varepsilon_k^i),\ 1 \le i \le m,$$
$$S_k^i(1) = S_k^i(0)(1 + \mu_k^i + \sum_{j=1}^m \kappa_k^i(j)\varepsilon_k^j + \bar{\kappa}_k^i \varepsilon_k^i),\ m < i \le k.$$

Here the μ_k^i are real numbers which can be interpreted as "expected return", the $\varepsilon_k^1, \dots, \varepsilon_k^m$ are the "random sources" driving the first m assets (which can be our focus assets or certain market indices); $\varepsilon_k^i,\ m+1 \le i \le k$ represents the randomness of S_k^i which is its own, the "idiosyncratic risk". From now on we suppose that the variables $\varepsilon_k^i,\ m+1 \le i \le k$ are independent for fixed k for all $k \ge m$ and that they are also independent from the family $\varepsilon_k^i,\ 1 \le i \le m$. We also assume $\bar{\kappa}_k^i \ne 0,\ k \ge m,\ i \le k$.

The parameter $\kappa_k^i(j)$ represents a certain "correlation" between market factor j and asset i in the kth small market. This is not to be taken in the formal sense as we do not suppose that the variables ε_k^i are square-integrable!

It is convenient to introduce a reparametrization.

$$S_k^0(0) = S_k^0(1) \equiv 1,$$
$$S_k^i(1) = S_k^i(0)(1 + \bar{\kappa}_k^i(\varepsilon_k^i - b_k^i)),\ 1 \le i \le m,$$
$$S_k^i(1) = S_k^i(0)(1 + \sum_{j=1}^m \kappa_k^i(j)(\varepsilon_k^j - b_k^j) + \bar{\kappa}_k^i(\varepsilon_k^i - b_k^i)),\ m < i \le k,$$

where

$$b_k^i = -\frac{\mu_k^i}{\bar{\kappa}_k^i},\ 1 \le i \le m, \quad b_k^i = -\frac{\mu_k^i}{\bar{\kappa}_k^i} + \sum_{j=1}^m \frac{\mu_k^j \kappa_k^i(j)}{\bar{\kappa}_k^j \bar{\kappa}_k^i},\ m < i \le k.$$

In the stationary case we drop double indices and write $\varepsilon_i, \kappa_i(j), \bar{\kappa}_i, S_i, b_i$ only. The APM fits well into the more general framework of large financial markets presented in the preceeding section.

In what follows we fix $1 \le \alpha \le 2$ and suppose that $\varepsilon_k^i,\ m < i \le k$ have standard symmetric α-stable distribution. That is, their characteristic function is given by

$$\phi(t) = e^{-|t|^\alpha}, \tag{3}$$

see [25] for a comprehensive treatment of these variables. The cases $\alpha = 1, 2$ correspond to the Cauchy and Gaussian distributions, respectively. What we are using is the following property: if X_1, X_2 are independent with distribution given by (3), then for $\sigma_1, \sigma_2 \in \mathbb{R}$

$$\sigma_1 X_1 + \sigma_2 X_2 \simeq \sqrt[\alpha]{|\sigma_1|^\alpha + |\sigma_2|^\alpha}\, X_1,$$

here $\simeq$ denotes equality in distribution. We know from [25] that the distribution of such variables is continuous and its support is the whole real line. In the sequel we will also need the conjugate number α' of α defined by

$$\frac{1}{\alpha} + \frac{1}{\alpha'} = 1.$$

Theorem 3.2 *If $\alpha > 1$ then (NAFL) implies that*

$$\sup_{k \ge m} \sum_{j=1}^k |b_k^j|^{\alpha'} < \infty. \tag{4}$$

In the case $\alpha = 1$

$$\sup_{k \ge m,\ j \le k} |b_k^j| < \infty. \tag{5}$$

If the model is stationary and the m*th small market satisfies the (NA) condition of Remark 2.8, we even have*

$$(EMMSS) \Longleftrightarrow (NAFL) \Longleftrightarrow \sum_{i=1}^\infty |b_i|^{\alpha'} < \infty.$$

for $\alpha > 1$ and

$$(EMMSS) \Longleftrightarrow (NAFL) \Longleftrightarrow \sup_{i \in \mathbb{N}} |b_i| < \infty$$

for $\alpha = 1$.

Remark 3.3 *The assumption (NA) is quite natural: if it fails, we can create arbitrage already in the* m*th market. From the proof it will be clear that if*

- *either* $\varepsilon_k^1, \dots, \varepsilon_k^m$ *are standard symmetric* α*-stable;*
- *or* $S_k^0, \dots, S_k^m$ *are the same variables for each* k *and (NA) holds in the market with these* $m+1$ *assets,*

then (4) *(resp.* (5)*) is also a sufficient condition for (NAFL), even in the non-stationary case. Finally, in the proof of sufficiency it is possible to drop the assumption that the factors are traded assets. It is also possible to extend the results for* $\alpha < 1$*. We only indicate these generalizations so as not to make the presentation complicated.*

Corollary 3.4 *If* $\alpha > 1$ *and there exists a constant* C *such that*

$$|\bar{\kappa}_i| \leq C, \quad i \in \mathbb{N},$$

then in the stationary model (NAFL) implies that there exists $\gamma_j,\ 1 \leq j \leq m$ *satisfying*

$$\sum_{i=m+1}^{\infty} \left| \mu_i - \sum_{j=1}^{m} \gamma_j \kappa_i(j) \right|^{\alpha'} < \infty. \tag{6}$$

Inequality (6) expresses that the expected return on asset k lies asymptotically close to a certain linear combination of the $\kappa_k(j),\ 1 \leq j \leq m$ as k tends to infinity; i.e. returns are "almost" linear functions of the "correlations" with the market factors.

In [9] a similar model was considered. The authors used a different concept of asymptotic arbitrage which depended on α. They found that condition (6) is necessary for the absence of "α-arbitrage". In the case $\alpha = 2$ (6) is identical to the condition given in [21].

4 Technicalities

We fix our probability space $(\Omega, \mathcal{F}, P)$ and we denote by L^0 the set of $\mathbb{R}$-valued random variables on it, L^0_+ is the set of non-negative random variables. As usual, we identify two random variables if they are equal almost surely. We equip L^0 with the (metrizable) topology of convergence in probability. For any $\tilde{P} \sim P$ we will denote by $L^1(\tilde{P})$ the Banach space of $\tilde{P}$-integrable functions, equipped with the usual norm topology.

We start with a useful sequential compactness result. If H is a subset of a vector space, conv(H) denotes the set of vectors which are finite convex combinations of elements of H.

Proposition 4.1 *Let f_n be a sequence in L^0 bounded from below by a random variable. Then there are $f'_n \in \mathrm{conv}(\{f_k, k \geq n\})$ which converge almost surely to some f with values in $\mathbb{R} \cup \{\infty\}$.*

Proof. This is trivial from Lemma A1.1 of [4]. □

Now we define an abstract analogue of the condition (NAFL). Let $K \subset L^0$ be a convex cone.

Definition 4.2 *We say that K has property (P), if there exists no sequence $h_n \in K$ with $h_n \to h$ in probability, $h \in F \setminus \{0\}$.*

If we take $K := C$ where C is defined by (1), we clearly have that K has (P) iff our stationary large financial market has (NAFL). The following lemma forms the main ingredient of the proof of Theorem 2.7. Notice that we do not use the specific market structure, only property (P) of K.

Lemma 4.3 *If K has property* (P) *then $\bar{K} - L^0_+$ is closed in probability. Here $\bar{K}$ is the closure of K in L^0.*

Proof. Let us suppose that

$$f_n - r_n \xrightarrow{n \to \infty} u,$$

in probability, where $f_n \in \bar{K}$, $r_n \in L^0_+$. By the definition of the closure of a set we may and will suppose that $f_n \in K$. Obviously,

$$f_n \geq f_n - r_n \geq \wedge_{n \in \mathbb{N}} (f_n - r_n) > -\infty$$

almost everywhere. By Proposition 4.1 we can take convex combinations f'_n of the f_n which converge to some f. We take the same convex combinations of r_n and denote them by r'_n. As K, L^0_+ are convex cones, we still have $f'_n \in K, r'_n \in L^0_+$. Let us define

$$V := \{\omega \in \Omega | f(\omega) = \infty\}.$$

We show that $P(V)$ is 0. Let us suppose that this is not true. Then we may find a subsequence n_k such that

$$P(V \cap \{f'_{n_k} > k^2\}) > P(V)(1 - \frac{1}{k}).$$

Then the sequence

$$g_k := \frac{f'_{n_k}}{k}$$

satisfies $g_k \in K$ and

$$g_k \xrightarrow{k\to\infty} \infty \text{ on } V, \quad g_k \xrightarrow{k\to\infty} 0 \text{ on } \Omega \setminus V,$$

in probability. As K has property (P), this is a contradiction. Now we may conclude that $f \in \bar{K}$, $r_n' \to r \in L^0_+$ and $u = f - r$, the Lemma is proved. □

For the construction of martingale measures we will use a version of the Kreps-Yan separation theorem (cf.[19, 26]).

Theorem 4.4 *Let $\tilde{P} \sim P$ and $D \subset L^1(\tilde{P})$ a closed convex cone with*

$$-L^1_+(\tilde{P}) \subset D, \quad D \cap L^1_+(\tilde{P}) = \{0\}.$$

Then there is an element $p \in L^\infty$ which is strictly positive and satisfies

$$\forall h \in D \; Eph \le 0.$$

Proof. See e.g. [22]. □

We also recall a useful result from p. 266 of [6].

Proposition 4.5 *If H is a countable set of random variables on $(\Omega, \mathcal{F}, P)$ then there exists $\tilde{P} \sim P$ with $d\tilde{P}/dP \in L^\infty$ such that each element of H is $\tilde{P}$-integrable.*

Now we are in the position to prove the main result of this article.

Proof of Theorem 2.7. First we assume (NAFL). We notice that (NAFL) is not sensible to an equivalent change of measure, so we may and will suppose $P = P'$. The cone C of (1) satisfies property (P), so by Lemma 4.3 $\bar{C} - L^0_+$ is closed in probability. Clearly $\bar{C} \cap L^0_+ = \{0\}$. We take a measure $\tilde{P} \sim P$, $d\tilde{P}/dP \in L^\infty$ which integrates all the S_k, $k \in \mathbb{N}$, hence all the elements of C. We apply Theorem 4.4 in $L^1(\tilde{P})$ to the closed convex cone

$$D := (\bar{C} - L^0_+) \cap L^1(\tilde{P}),$$

and obtain an element $p \in L^\infty$ which defines a measure $Q \sim \tilde{P}$ by setting

$$\frac{dQ}{d\tilde{P}} := \frac{p}{Ep}.$$

As $E^Q \pm (S_k(1) - S_k(0)) \le 0, \; k \in \mathbb{N}$, we get

$$E^Q(S_k(1) - S_k(0)) = 0, \quad k \in \mathbb{N},$$

i.e. Q is a martingale measure for this market.

The other direction: if we have a sequence

$$d_n \xrightarrow{n\to\infty} d, \; d_n \in C, \; n \in \mathbb{N},$$

violating (NAFL), we may suppose that the convergence is almost everywhere. We introduce $P' \sim P$ which integrates the lower envelope

$$\wedge_{n\in\mathbb{N}} d_n > -\infty.$$

Taking a martingale measure Q for P' with $dQ/dP' \in L^\infty$ we get that

$$E^Q d_n = 0, \; n \in \mathbb{N}.$$

Using Fatou's lemma we find that

$$E^Q d \leq 0,$$

in contradiction with $d \in F \setminus \{0\}$.

Before proceeding to the proof of Theorem 3.2 we make a few useful observations.

Proposition 4.6 *Let $X_n, \; n \in \mathbb{N}$ be a sequence of identically distributed (but not necessarily independent!) random variables such that their common distribution is continuous and its support is the whole real line. Let σ_n be a sequence of positive numbers, μ_n a sequence of real numbers. If*

$$\frac{\mu_n}{\sigma_n} \to \infty, \; \mu_n \to \infty, \; n \to \infty \tag{7}$$

then $\sigma_n X_n + \mu_n \to \infty$ in probability. If

$$\sup_{n\in\mathbb{N}} \left| \frac{\mu_n}{\sigma_n} \right| < \infty \tag{8}$$

then either there is $\beta > 0$ such that $P(\sigma_n X_n + \mu_n \leq -1) \geq \beta$ or

$$\sigma_n X_n + \mu_n \to 0$$

almost surely along a suitable subsequence.

Proof. If (7) holds then

$$\forall K > 0 \; P(\sigma_n X_n + \mu_n > K) = P(\frac{\sigma_n}{\mu_n} X_n + 1 > \frac{K}{\mu_n}) \to P(1 > 0) = 1.$$

If one has (8) then we distinguish two cases. If $\sup_n \sigma_n$ is finite then $\sup_n |\mu_n|$ is finite, too. Taking a subsequence we may suppose that the two sequences converge to σ, μ respectively. If $\sigma = 0$ then $\mu = 0$ and we are done. If not, we have

$$P(\sigma_n X_n + \mu_n \leq -1) \to P(\sigma X_1 + \mu \leq -1) > 0$$

by the assumption on the distributions. If $\sup_n \sigma_n = \infty$ then we take a subsequence (still denoted by n) such that $\sigma_n \to \infty$ and $\mu_n/\sigma_n \to c \in \mathbb{R}$. We get

$$P(\sigma_n X_n + \mu_n \leq -1) = P(X_n + \frac{\mu_n}{\sigma_n} + \frac{1}{\sigma_n} \leq 0) \to P(X_1 \leq -c) > 0.$$

$\square$

We introduce the following notation: if $Y \simeq \sigma X + \mu$ where X is standard symmetric α-stable then we will write $\sigma(Y), \mu(Y)$ for σ, μ, respectively.
Proof of Theorem 3.2. (NAFL)$\Leftrightarrow$(EMMSS) follows from the main result in section 2. We only have to concentrate on the other implications.

Necessity: We take ℓ^α, the Banach space of sequences $(l_n)_{n\in\mathbb{N}}$ satisfying

$$\|l\|_\alpha := \sum_{n=1}^{\infty} |l_n|^\alpha < \infty.$$

We fix any element $0 \neq l \in \ell^\alpha$. We define the strategy ϕ_k on the kth market for all $k > m$:

$$\phi_k^i := \frac{\text{sgn}(l_i b_k^i) l_i}{\bar{\kappa}_i^k}, \ m < i \leq k;$$
$$\phi_k^i := \frac{\text{sgn}(l_i b_k^i) l_i - \sum_{j=m+1}^k \phi_k^j \kappa_k^j(i)}{\bar{\kappa}_i^k}, \ 1 \leq i \leq m.$$

It is easy to see that in this case V^{ϕ_k} is an α-stable variable. Direct calculation gives its scale and shift parameter. We notice that

$$\sigma(V^{\phi_k}) \to \|l\|_\alpha \neq 0. \tag{9}$$

(NAFL) holds, so by Proposition 4.6 we conclude that

$$\sup_{k>m} |\mu(V^{\phi_k})| = \sup_{k>m} \sum_{i=1}^{k} |l_i b_k^i| < \infty. \tag{10}$$

We may consider the sequences $v_k = v_k^i, k > m$ defined by

$$v_k^i = b_k^i, \ 1 \leq i \leq k, \quad v_k^i = 0, \ i > k$$

as elements of $\ell^{\alpha'}$, the dual space of ℓ^{α}. We may interpret (10) as follows: the sequence v_k of continuous linear functionals on ℓ^{α} is pointwise bounded. So, by the Banach-Steinhaus theorem it is bounded in norm, which is just (4). If $\alpha = 1$ then $\ell^{\alpha'}$ is the space of bounded sequences ℓ^{∞} with the supremum norm and we obtain (5).

Sufficiency in the stationary case: Let us suppose that there is a sequence of portfolios $\phi_k,\ k > m$ such that

$$V^{\phi_k} \to V \in F \setminus \{0\}. \tag{11}$$

We cut V^{ϕ_k} in two:

$$V^{\phi_k} = V_1^k + V_2^k := \sum_{i=1}^{m} \left(\phi_i + \sum_{j=m+1}^{k} \phi_j \kappa_j(i) \right) \varepsilon_i + \sum_{i=m+1}^{k} \phi_i \bar{\kappa}_i \varepsilon_i.$$

Using the Hölder-inequality we get

$$\begin{aligned} \left| \frac{\mu(V_2^k)}{\sigma(V_2^k)} \right| &= \left| \frac{-\sum_{i=1}^{m} (\phi_i + \sum_{j=m+1}^{k} \phi_j \kappa_j(i)) b_i - \sum_{i=m+1}^{k} \phi_i \bar{\kappa}_i b_i}{\sqrt[\alpha]{\sum_{i=1}^{m} |\phi_i + \sum_{j=m+1}^{k} \phi_j \kappa_j(i)|^{\alpha} + \sum_{i=m+1}^{k} |\phi_i \bar{\kappa}_i|^{\alpha}}} \right| \\ &\leq \sqrt[\alpha']{\sum_{i=1}^{k} |b_i|^{\alpha'}} \leq \|b\|_{\alpha'}, \end{aligned}$$

for $\alpha > 1$ and

$$\left| \frac{\mu(V_2^k)}{\sigma(V_2^k)} \right| \leq \|b\|_{\infty},$$

for $\alpha = 1$. We infer from Proposition 4.6 that

$$\forall k\ P(V_2^k \leq -1) \geq \beta > 0, \tag{12}$$

for some β or $V_2^k \to 0$ almost surely (along a subsequence). The latter case is easy to handle, so we suppose (12) and turn our attention to V_1^k. If there is a $\theta > 0$ such that

$$\forall k\ P(V_1^k \leq -\theta) \geq \theta,$$

then we are done, as by independence

$$P(\{V_1^k \leq -\theta, V_2^k \leq -1\}) \geq \theta\beta > 0,$$

for all k, a contradiction to (11). If this is not the case then the negative part of a subsequence (still denoted by k) converges to 0 almost surely:

$$(V_1^k)^- \to 0.$$

Using the property (NA) and Remark 2.7 we may conclude that V_1^k converges to 0 and (12) contradicts (11).
Proof of Corollary 3.4. Under the hypotheses we get that

$$\sum_{i=m+1}^{\infty} \left| \mu_i - \sum_{j=1}^{m} \mu_j \frac{\kappa_i(j)}{\bar{\kappa}_i} \right|^{\alpha'} < \infty,$$

so we can set $\gamma_j := \mu_j / \bar{\kappa}_j$.

5 Conclusion

In this article we were dealing with certain problems of the arbitrage theory of discrete-time large financial markets. We have shown that there is no sequence of portfolios whose returns converge to a strictly positive (possibly infinite) random variable in probability if and only if for each measure $P' \sim P$ there is an equivalent martingale measure Q with bounded dQ/dP'. We suggested an economic interpretation of this criterion. In a certain class of models we have given explicit characterizations in terms of market parameters.

One would naturally ask what is the right asymptotic no-arbitrage concept which is equivalent to (EMM) (and does not create a whole lot of martingale measures like in (EMMSS)). It would also be desirable to characterize whether the classical APM admits an equivalent martingale measure in the case where the ε_i have arbitrary distributions. Results of [20] show that (under certain conditions on the ε_i) this follows from the by now familiar condition

$$\sum_{k=1}^{\infty} b_k^2 < \infty.$$

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