Log-optimal currency portfolios and control Lyapunov exponents

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A FAIR GAME

Example 1: Let the price of a stock at time t be S_t. Assume the price dynamics

$$S_{t+1} = \begin{cases} 1.2\ S_t & \text{w.p. } 1/2, \\ 0.8\ S_t & \text{w.p. } 1/2. \end{cases}$$

Thus

$$\mathrm{E}S_t = \text{const.}$$

A FAVORABLE GAME

Example 2:

$$S_{t+1} = \begin{cases} 1.24\ S_t & \text{w.p. } 1/2 \\ 0.8\ S_t & \text{w.p. } 1/2. \end{cases}$$

Now we have

$$\mathsf{E} S_{t+1} = 1.02 \cdot \mathsf{E} S_t.$$

Thus

$$\lim \mathsf{E} S_T = 0.$$

LONG TERM PROSPECTS

Write

$$S_{t+1} = Y_{t+1} S_t.$$

Then

$$S_T = Y_T ... Y_1 \cdot S_0.$$

Assume that (Y_t) is i.i.d., then w.p.1

$$\lim \frac{\log S_T}{T} = \mathsf{E} \log Y.$$

This is the growth rate of the wealth process.

GROWTH RATE FOR EXAMPLE 2

We have

$$\log \mathsf{E}Y = \frac{1}{2}(\log 1.24 + \log 0.8) = \frac{1}{2}(\log 0.92) < 0!$$

Thus w.p.1

$$\lim \log S_T = 0 \ !$$

A LOG-FAIR PRICE DYNAMICS

Assume

$$S_{t+1} = Y_{t+1} S_t,$$

where Y_t is i.i.d. The stock is log-fair if

$$\mathsf{E} \log Y = 0.$$

Then the growth-rate is zero.

EXAMPLE 3

Let

$$Y_t = \begin{cases} 1.25 & \text{w.p. } 1/2, \\ 0.8 & \text{w.p. } 1/2. \end{cases}$$

In general let

$$Y_t = \begin{cases} c > 1 & \text{w.p. } 1/2, \\ c^{-1} & \text{w.p. } 1/2. \end{cases}$$

COVER

A market with m stocks: the price of stock i on day t is

$$S_{i,t}.$$

The price-dynamics:

$$S_{i,t+1} = Y_{i,t+1} S_{i,t}$$

The bond price: $S_{0,t} = 1$ for all t.

The returns $Y_{i,t+1}$ are random, i.i.d. Set

$$\mathbf{Y}_t = (Y_{1,t}, \ldots, Y_{m,t}).$$

A RELATIVE PORTFOLIO

On the morning of day t: invest a fraction $b_{i,t}$ of the total wealth in stock i. Thus

$$b_{0t} + \cdots + b_{mt} = 1 \qquad b_{it} \geq 0.$$

Set

$$\mathbf{b}_t = (b_{0,t}, \ldots b_{m,t}).$$

At the end of day t: the total wealth gets multiplied by

$$\sum_i Y_{it}\ b_{it}.$$

The wealth on day n is

$$W_n = W_0 \prod_{i=1}^{n} \langle \mathbf{b}_t, \mathbf{Y}_t \rangle.$$

LOG-OPTIMAL PORTFOLIOS

Problem: find an optimal portfolio. For a fixed b the growth-rate of W_n is:

$$\frac{1}{n}\sum_{t=1}^{n} \log\langle \mathbf{b}, \mathbf{Y}_t \rangle.$$

If $(\mathbf{Y}_i)$ is i.i.d then w.p.1 the limit is

$$\mathsf{E}\log\langle \mathbf{b}, \mathbf{Y} \rangle.$$

The log-optimal portfolio: maximize the above growth-rate in b.

EXAMPLE: A LOG-FAIR STOCK

Assume $m = 1$ and

$$Y_t := \begin{cases} c > 1, & \text{w. p. } 1/2 \\ c^{-1}, & \text{w. p. } 1/2. \end{cases}$$

Then the log-optimal portfolio is:

$$b^* = (1/2, 1/2).$$

The optimal growth-rate is always positive! For $c = 2$ we have $(9/8)^{1/2} > 1$.

EXAMPLE: A FAIR STOCK

Let $m = 1$ and

$$Y_t = \begin{cases} 1.2 & \text{w.p. } 1/2, \\ 0.8 & \text{w.p. } 1/2. \end{cases}$$

By Jensen's inequality

$$\mathsf{E} \log \langle \mathbf{b}, \mathbf{Y} \rangle \leq \log \mathsf{E} \langle \mathbf{b}, \mathbf{Y} \rangle = 0$$

since

$$\mathsf{E}\mathbf{Y_t} = (1, 1).$$

Thus the optimal growth-rate is always non-positive !

EXAMPLE: A FAVORABLE STOCK

Let

$$Y_t = \begin{cases} 1.2 + \delta & \text{w.p. } 1/2, \\ 0.8t & \text{w.p. } 1/2. \end{cases}$$

with $\delta > 0$.

Then optimal growth-rate is positive !

A CURRENCY MARKET

A market with friction: let us consider k currencies. Let at any time t

$$p_{ij}$$

be the exchange rate. Write also $p_{ij} = p_{ij,t}$.

Thus p_{ij} is the number of units in currency i which can be purchased for 1 unit of currency j.

Define the exchange rate matrix as

$$P_t = (p_{ij,t}).$$

A CURRENCY MARKET

Assuming proportional transaction costs we have

$$p_{ij}p_{jl} \leq p_{il}.$$

Example: A friction-less market:

$$p_{ij} = S_j/S_i.$$

REBALANCING

At any time t let

$$\phi_t = (\phi_{i,t})$$

denote the absolute (physical) portfolio. Let

$$\beta_{ij}$$

be the fraction of wealth in currency i (evaluated in a *fixed* currency) which is transferred to currency j.

Obviously for any $i \neq j$ we have

$$\beta_{ij} \cdot \beta_{ji} = 0.$$

Set

$$B_t = (\beta_{ij,t}).$$

THE PORTFOLIO DYNAMICS

Let the current portfolio be

$$\phi = (\phi_1, \ldots, \phi_k).$$

Then the amount of currency i at the next period is

$$\phi_i^+ = \sum_{j=1}^{k} p_{ij}\beta_{ij}\phi_j.$$

Write for the matrix with elements $(p_{ij}\beta_{ij})$

$$X = P \odot B.$$

Then the dynamics for the physical portfolio is

$$\phi_{t+1} = (P_{t+1} \odot B_t)\phi_t.$$

This is a random linear dynamics, controlled by (B_t).

THE WEALTH PROCESS

Consider parametric strategies

$$B = (B_t(\theta)).$$

The value of the portfolio in, say, currency i is

$$V_{i,t} = \sum_j p_{ij,t}\phi_{j,t}.$$

Problem: Maximize the growth-rate

$$\lambda(\theta) := \lim_{t\to\infty} \frac{1}{t} \log V_t(\theta).$$

Remark: The growth rate of the wealth process is independent of the numeraire.

RANDOM MATRIX-PRODUCTS

Let

$$X = (X_n), n = 0, 1, ...$$

be a stationary, ergodic process of $k \times k$ real-valued matrices, satisfying

$$\mathsf{E} \log^+ ||X_0|| < \infty.$$

Define the top Lyapunov-exponent of the process (X_n) as

$$\lambda = \lim_n \frac{1}{n} \mathsf{E} \log ||X_n \cdot X_{n-1} \cdots X_0||.$$

It is the exponential growth rate of the product $||X_n \cdot X_{n-1} ... \cdot X_0||$.

RANDOM MATRIX-PRODUCTS

Theorem 1. (Fürstenberg-Kesten, 1960). *We have almost surely*

$$\lambda = \lim_n \frac{1}{n} \log ||X_n \cdot X_{n-1} \cdots X_0||.$$

RANDOM MATRIX-PRODUCTS

Question: what can we say about the vector-norms

$$\lambda = \lim_n \frac{1}{n} \log |X_n \cdot X_{n-1} \cdots X_0 v|$$

for any v ?

OSELEDEC'S THEOREM

Theorem 2. (Oseledec, 1968) There exists a $\Omega' \subset \Omega$ with $P(\Omega') = 1$ and a random subspace of fixed dimension $V_1(\omega) \subset \mathbb{R}^k$, such that for

$$\omega \in \Omega', \qquad v \in \mathbb{R}^k \setminus V_1(\omega)$$

we have

$$\lim_n \frac{1}{n} \log |X_n(\omega) X_{n-1}(\omega) \cdots X_1(\omega) v| = \lambda.$$

A RANK 1 PROCESS

Let

$$X_n = u_n v_n^T,$$

with $|u_n| = |v_n| = 1$, where (u_n, v_n) is strictly stationary.

Then $X_n \dots X_0 = u_n(v_n^T u_{n-1}) \dots (v_1^T u_0) v_0^T$.

Thus the top Lyapunov-exponent is

$$\lambda = \mathsf{E} \log(v_1^T u_0).$$

The exceptional subspace: if

$$v_0^T \xi = 0$$

then

$$X_n \dots X_0 \xi = 0$$

for all n.

CONTROLLED LYAPUNOV EXPONENTS

Assume that

$$X_n = X_n(\theta),$$

where $\theta \in D \subset \mathbb{R}^p$, is a control-parameter.

Thus the top Lyapunov-exponent $\lambda = \lambda(\theta)$ will be a function of θ.

Problem. Find

$$\max_\theta \lambda(\theta).$$

A POLAR DECOMPOSITION

Let $\xi \notin V_1(\omega)$, $\omega \in \Omega'$, and $\lambda > -\infty$ define the normalized process

$$z_n = X_n \cdot X_{n-1} \cdots X_1 \xi / |X_n \cdot X_{n-1} \cdots X_1 \xi|.$$

A recursion for the normalized process z_n:

$$z_{n+1} = X_{n+1} z_n / |X_{n+1} z_n|$$

or

$$z_{n+1} = \Pi_v(X_{n+1} z_n).$$

where

$$\Pi_v(y) = \frac{y}{|y|}.$$

THE TOP LYAPUNOV EXPONENT

We can write λ as an arithmetic mean:

$$\lambda = \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \log |X_{k+1} z_k|. \qquad (1)$$

A RANK-1 APPROXIMATION

Proposition 1 *Assume that the co-dimension of $V_1(\omega)$ is 1. Then*

$$Z_n = u_n^* v_0^T + O_M(e^{-\epsilon n}),$$

where (u_n^*) is strictly stationary.

EXISTENCE OF STATIONARY INITIALIZATION

Using a well-known backward construction we get:

Proposition 2 *Let V_1 have co-dimension 1. Let ξ be uniformly distributed over the unit sphere and let it be independent of (X_n) and define*

$$z_0^* = \lim_n \Pi_v(X_0 X_{-1} \dots X_{-n}\xi).$$

Then z_0^* is a stationary initialization.

ANY INITIALIZATION

The process (X_n, z_n) is asymptotically stationary for any good initialization:

Proposition 3 *Let V_1 have co-dimension 1. For any initialization $z_0 = z_0(\omega)$ such that*

$$z_0(\omega) \notin V_1(\omega)$$

for $\omega \in \Omega'$ we have

$$z_n = z_n^* + \delta z_n,$$

where

$$|\delta z_n| \leq C(\omega) e^{-\gamma n}$$

where $C(\omega)$ is finite and $\gamma > 0$ is a constant.

STABILITY WITH RESPECT TO INITIAL VALUE

Proposition 4 *Let V_1 have co-dimension 1. Assume that*

$$\xi = z_0(\omega) \notin V_1(\omega)$$

for $\omega \in \Omega'$. Then

$$\|\frac{\partial z_n}{\partial \xi}\| \leq C(\omega)e^{-\gamma n}$$

where $C(\omega, \xi)$ is finite and $\gamma > 0$ is a constant.

Extends the results of:

Atar and Zeitouni, SIAM J. Contr. Opt., 1996.

THE GRADIENT OF Xz

Write

$$\frac{d}{dt}\text{log } |X(t)z(t)| =$$

$$= \frac{1}{|Xz|^2}(z^T X^T X \dot{z} + z^T X^T \dot{X} z =$$

$$= \dot{H}(X, z, \dot{X}, \dot{z}).$$

THE GRADIENT OF z_k

Consider the mapping

$$f(X, z) = Xz/|Xz|$$

with $Xz \neq 0$. Write

$$\frac{d}{dt} f(X(t), z(t)) = f_X \dot{X} + f_z \dot{z} = g(X, z, \dot{X}, \dot{z}).$$

UPDATING θ

At time n set

$$\begin{aligned} z_{n+1} &= X_{n+1} z_n / \|X_{n+1} z_n\| \\ z_{\theta_i, n+1} &= g(X_{n+1}, z_n, X_{\theta_i, n+1}, z_{\theta_i, n}) \\ H_n &= H(X_{n+1}, X_{\theta, n+1}, z_n, z_{\theta, n}) \end{aligned}$$

$$\theta_{n+1} = \theta_n - \frac{1}{n} H_n.$$

Convergence analysis:

The i.i.d case: use the theory of Benveniste, Metivier and Priouret.

The general case: extend the so-called ODE method.

STATE DEPENDENT RANDOM PRODUCTS

Consider the operation

$$X = P \odot B(\phi)$$

where $B(\phi)$ is the redistribution matrix.

Note that $B(\phi)$ depends on the current portfolio ϕ. Thus we get

$$X_{n+1} = P_{n+1} \odot B(\phi_n)$$

$$\phi_{n+1} = X(P_{n+1}, \phi_n)\phi_n.$$

Note that $B(\phi)$ is scale-invariant, thus

$$X(P, \phi) = X(P, z)$$

$$z = \phi/|\phi|.$$

TWO CURRENCIES

Take a log-fair market with friction, $k = 2$, and exchange rate dynamics

$$p_{12}(t+1) = p_{12}(t) Y_{t+1}(1 - d\varepsilon_{t+1}),$$
$$p_{21}(t+1) = p_{12}(t) \frac{1}{Y_{t+1}}(1 - d\varepsilon_{t+1}),$$

with (ε_t) i.i.d. uniform on $[0, 1]$, $0 \leq d < 1$.

As usual: Y_t are i.i.d.

$$P(Y_t = 2) = P(Y_t = 1/2) = 1/2.$$

THE EFFECT OF FRICTION

The frictionless case: $d = 0$. Get $\alpha^* = 0.5$ and

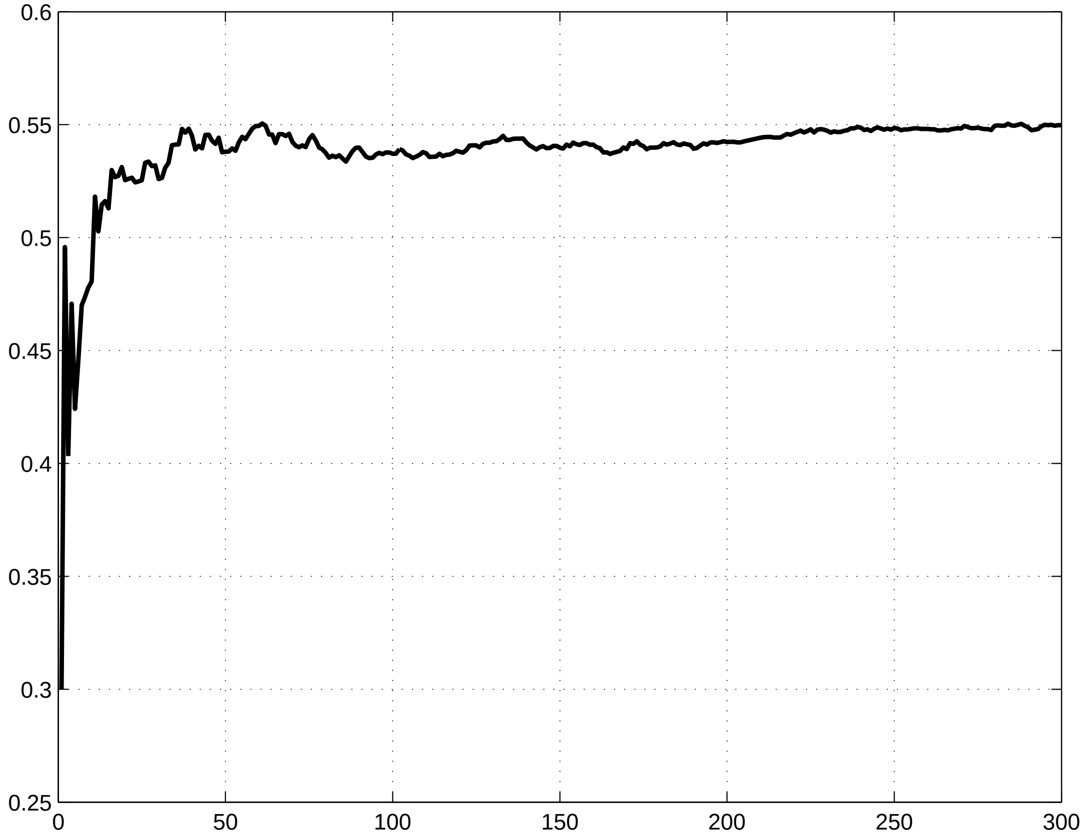
$$\lambda(\alpha^*) = 0.0626.$$

A market with friction: $d = 0.5$. Get $\alpha^* = 0.54$ and

$$\lambda(\alpha^*) = 0.04861.$$

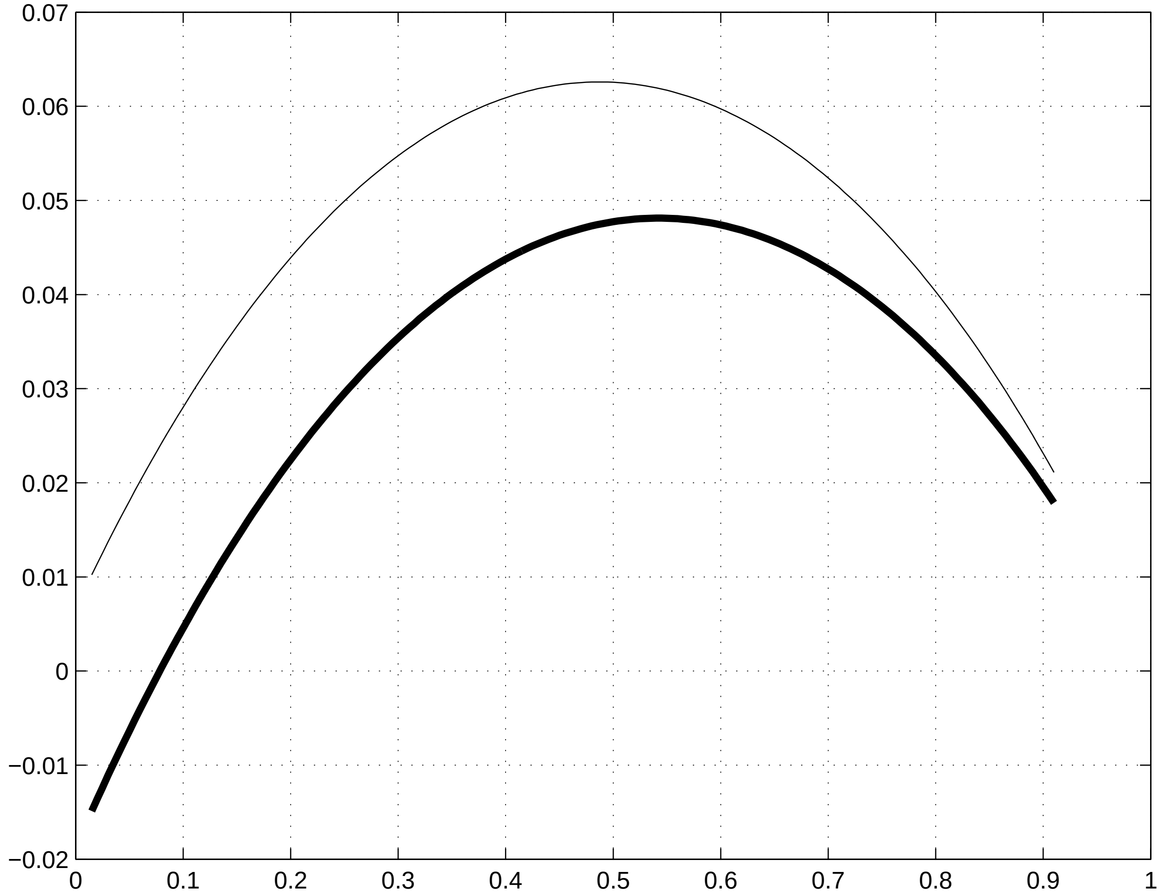
A 5% friction causes cca 22% loss in rate !

ITERATION ON α



Convergence to $\alpha^* = 0.54$ with initialization $\alpha_0 := 0.3$.

THE TOP LYAPUNOV EXPONENT AS A FUNCTION OF α



Convergence to $\alpha^* = 0.54$ with initialization $\alpha_0 := 0.3$.

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