# BIBO Stability of Linear Switching Systems 

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#### Abstract

In this note, we show that for linear switching systems of the form $x_{n+1}=A_{n} x_{n}+u_{n+1}$, where the matrices $A_{n}$ are chosen arbitrarily from a given set of matrices, bounded-input-bounded-output stability implies uniform exponential stability.


Index Terms-Bounded-input-bounded-output (BIBO) stability, switching systems, uniform exponential stability.

## I. Introduction

Suppose that we are given a set of real $p \times p$ matrices $\mathcal{A}=\left\{A_{\gamma} \mid \gamma \in \Gamma\right\}$, where $\Gamma$ is an arbitrary set of indexes. Consider the switching linear system

$$
\begin{equation*}
x(n+1)=A_{\sigma(n)} x(n)+u(n+1) \tag{1.1}
\end{equation*}
$$

where $x(0) \in \mathbb{R}^{p}, \sigma:[0,1, \ldots) \rightarrow \Gamma$ is an arbitrary switching signal, while $u(1), u(2), \ldots, \in \mathbb{R}^{p}$ is an input signal. The problem of finding conditions for various kinds of stability of the system above has received a fair amount of attention recently, both for discrete-time and continuous-time systems.

A number of papers focused on finding appropriate switching strategy in order to stabilize the system; see, e.g., [9], [10], and [15]. Another line of research is to characterize systems which are asymptotically stable for any arbitrary switching signal $\sigma$; see, e.g., [5], [7], [8], [12], and [13]. In [1] and [6], Lie-algebraic conditions are given implying the existence of a common quadratic Lyapunov-function. Dayawansa and Martin [3] proved that for compact linear polysystems uniform asymptotic stability implies the existence of a common Lyapunov function, which is not necessarily quadratic. Their proof is carried out for continuous-time systems, but without too much effort it works for discrete-time systems, as well.

Stability of switching systems is an important technical issue in recursive identification and stochastic adaptive control. In the context of recursive identification, the estimated inverse system is time varying and the set of inverse systems should be restricted so that the stability of the time varying system is ensured. Assuming that the inverse plants can be described by finite-dimensional systems with fixed dimension the problem reduces to finding conditions under which the response of a time-varying linear system exited by a signal which is bounded in a certain stochastic sense, stays bounded in a certain stochastic sense. Such a condition is given in [4, Condition 4.2], where the existence of a joint quadratic Lyapunov function is assumed. In the light of the results of the work of [3], this condition is obviously too restrictive. The question arises as to what extent can [4, Condition 4.2] be relaxed. Similar problems arise in stochastic adaptive control, where the controller

[^0]is time-varying, and the set of controllers should be restricted so that the stability of the time-varying closed-loop system is ensured for any choice of the controller.

In this note, the following closely related result will be proved: assume that for any arbitrary switching path $\sigma$ the linear time-varying system (1.1) is bounded-input-bounded-output (BIBO) stable in the sense that for any bounded input sequence $u(j), j \geq 1$, the output sequence $x(j), j \geq 0$ is bounded, as well. Then the homogeneous linear switching-system is exponentially stable in a sense to be defined later.

## II. BIBO Stability

Denote by $\mathbb{R}_{\infty}^{p}$ the set of infinite sequences of $p$-dimensional vectors, and by $l_{\infty}^{p}$ the set of norm-bounded infinite sequences of $p$-dimensional vectors.

For any switching path $\sigma:[0,1, \ldots) \rightarrow \Gamma$, let us denote by $T_{\sigma}$ the linear operator for which

$$
\begin{equation*}
T_{\sigma}(x(0), u(1), u(2), \ldots)=(x(0), x(1), x(2), \ldots) \tag{2.1}
\end{equation*}
$$

where the sequence $x(j), j \geq 0$ is generated by

$$
x(n+1)=A_{\sigma(n)} x(n)+u(n+1)
$$

from the initial vector $x(0)$. Obviously

$$
T_{\sigma}: \mathbb{R}_{\infty}^{p} \rightarrow \mathbb{R}_{\infty}^{p}
$$

Note that introducing the initial condition $x(-1)=0$ we obtain that $x(0)$ can be considered as $u(0)=x(0)$. Using this, $T_{\sigma}$ can be written shortly as

where $u=(u(0), u(1), \ldots), x=(x(0), x(1), \ldots)$.
Definition 2.1: The switching system (1.1) determined by the set

$$
\mathcal{A}=\left\{A_{\gamma} \mid \gamma \in \Gamma\right\} \subset \mathbb{R}^{p \times p}
$$

is BIBO stable, if, for any switching path $\sigma$, the linear operator $T_{\sigma}$ maps bounded input sequences into bounded output sequences. In other words

$$
T_{\sigma}\left(l_{\infty}^{p}\right) \subset l_{\infty}^{p}
$$

Definition 2.2: The switching system (1.1) is uniformly exponentially stable, if there exist a $\lambda<1$ and $c<\infty$, such that for any switching path $\sigma$ and for the identically zero input $u(j)=0, j \geq 1$, the norm of the output sequence $x(n), n \geq 0$ defined by the (1.1) can be bounded above as follows:

$$
\|x(n)\| \leq c \lambda^{n}\|x(0)\|
$$

Uniform BIBO stability of linear, time-varying systems are analyzed, for example, by Rugh [11] in the continuous-time case. Some of the results of [11] could be used here to make the proofs a bit shorter, but for sake of convenience complete, self-contained proofs are presented here.

Theorem 2.1: The switching system (1.1) is BIBO stable if and only if it is uniformly exponentially stable.

Proof: The uniform exponential stability obviously implies the BIBO stability due to the trivial upper bound

$$
\|x(n)\| \leq\left\|A_{\sigma(n-1)} A_{\sigma(n-2)} \cdots A_{\sigma(0)} x(0)\right\|
$$

$$
\begin{aligned}
& +\sum_{j=1}^{n}\left\|A_{\sigma(n-1)} \cdots A_{\sigma(j)} u(j)\right\| \\
\leq & c \lambda^{n}\|x(0)\|+\sum_{j=1}^{n} c \lambda^{n-j}\|u(j)\| \\
\leq & c\left(\|x(0)\|+\frac{\lambda}{1-\lambda} \sup _{j \geq 1}\|u(j)\|\right) \\
\leq & c \frac{1}{1-\lambda} \sup _{j \geq 0}\|u(j)\|
\end{aligned}
$$

for all $n \geq 0$.
Conversely, let us assume that the system (1.1) is BIBO stable. First, we show that for any fixed switching path $\sigma$ the operator $T_{\sigma}$ is bounded on the Banach-space $l_{\infty}^{p}$. In fact, due to its finite structure, i.e., the value of $x(n)$ depends only on the values of finitely many input vectors, namely on $u(1), \ldots, u(n)$ and the initial condition $x(0)$, it is easy to see that it is a closed operator. Since it is defined on the whole Banachspace $l_{\infty}^{p}$ it should be continuous. Denote its norm by

$$
K_{\sigma}=\left\|T_{\sigma}\right\|
$$

Now, fix a switching path $\sigma$ and an initial condition $x(0)$ and define the input $u(j)=0$, if $j \geq 1$. Denote the corresponding output sequence by $x^{\sigma}(n)$

$$
\begin{aligned}
x^{\sigma}(0) & =x(0) \\
x^{\sigma}(n) & =A_{\sigma(n-1)} A_{\sigma(n-2)} \cdots A_{0} x(0) .
\end{aligned}
$$

Then, the inequality

$$
\left\|x^{\sigma}(n)\right\| \leq K_{\sigma}\|x(0)\|
$$

holds. Thus, $x^{\sigma}(n)$ is a bounded sequence. Let us use this sequence as a new input sequence, in other words apply the operator $T_{\sigma}$ again. We obtain again a bounded sequence, thus the operator $T_{\sigma}$ can be applied again. Therefore, the following recursive definition will be used:

$$
\begin{align*}
z_{0}^{\sigma}(0) & =x(0)  \tag{2.2}\\
z_{0}^{\sigma}(n) & =x^{\sigma}(n)  \tag{2.3}\\
z_{k}^{\sigma}(0) & =x(0)  \tag{2.4}\\
z_{k}^{\sigma}(n+1) & =A_{\sigma(n)} z_{k}^{\sigma}(n)+z_{k-1}^{\sigma}(n+1) \\
k & \geq 1, \quad n \geq 0 . \tag{2.5}
\end{align*}
$$

Since the sequence $\left(z_{k}^{\sigma}(0), z_{k}^{\sigma}(1), z_{k}^{\sigma}(2), \ldots\right)$ is obtained from the sequence $(x(0), 0,0, \ldots)$ applying the operator $T_{\sigma} k+1$ times, i.e.,

$$
z_{k}^{\sigma}=\left(T_{\sigma}\right)^{k+1} x
$$

where $z_{k}^{\sigma}=\left(z_{k}^{\sigma}(0), z_{k}^{\sigma}(1), \ldots\right), x=(x(0), 0,0, \ldots)$ it follows that

$$
\begin{equation*}
\left\|z_{k}^{\sigma}(n)\right\| \leq\left(K_{\sigma}\right)^{k+1}\|x(0)\| \tag{2.6}
\end{equation*}
$$

The following lemma will be used to set a lower bound for $z_{k}^{\sigma}(n)$. Lemma 2.1: We have

$$
\begin{equation*}
z_{k}^{\sigma}(n)=\binom{k+n}{k} x^{\sigma}(n) \tag{2.7}
\end{equation*}
$$

Proof: We prove the lemma by induction with respect to $k$ and $n$. For $k=0$ and $n$ arbitrary (2.7) holds by definition. Similarly, if $n=0$ and $k$ is arbitrary (2.7) holds again.

Now, let us fix $k$ and $n$, and assume that (2.7) is true for $z_{j}^{\sigma}(l)$, if $j<k$ and $l \geq 0$ and if $j=k$ and $l \leq n$. We are going to show that it holds for $z_{k}^{\sigma}(n+1)$, as well. In fact

$$
z_{k}^{\sigma}(n+1)=A_{\sigma(n)} z_{k}^{\sigma}(n)+z_{k-1}^{\sigma}(n+1)
$$

$$
\begin{aligned}
= & \binom{k+n}{k} A_{\sigma(n)} x^{\sigma}(n) \\
& +\binom{k-1+n+1}{k-1} x^{\sigma}(n+1) \\
= & \binom{k+1+n}{k} x^{\sigma}(n+1)
\end{aligned}
$$

using the identities $x^{\sigma}(n+1)=A_{\sigma(n)} x^{\sigma}(n)$ and $\binom{n+k}{k}+$ $\binom{n+k}{k-1}=\binom{n+k+1}{k}$. Keeping $k$ fixed it follows from the induction step that (2.7) holds for all $n$. Now, change $k$ to $k+1$. Equation (2.7) holds for $n=0$, consequently the induction step with respect to $n$ can be applied again. This concludes the proof of the lemma.
From (2.6) and (2.7), it follows that

$$
\begin{equation*}
\left\|x^{\sigma}(n)\right\| \leq \frac{\left(K_{\sigma}\right)^{k+1}}{\binom{k+n}{k}}\left\|x_{0}\right\| \tag{2.8}
\end{equation*}
$$

for any $k \geq 0$. Consequently, the sequence $\left\|x^{\sigma}(n)\right\|$ tends to zero with more than any polynomial speed. We show that the rate of convergence is in fact exponentially fast.

Note that (2.8) can be expressed in the following manner:

$$
\begin{equation*}
\left\|A_{\sigma(n-1)} A_{\sigma(n-2)} \cdots A_{\sigma(0)}\right\| \leq \frac{\left(K_{\sigma}\right)^{k+1}}{\binom{k+n}{k}} \tag{2.9}
\end{equation*}
$$

for any $k \geq 0$.
Now, working along the same switching path $\sigma$, consider a vector $x \in \mathbb{R}^{p}$ and fix an index $j \geq 1$. Excite (1.1) by the impulse

$$
u_{j, x}(n)= \begin{cases}0, & \text { if } n \neq j \\ x, & \text { if } n=j\end{cases}
$$

Observe that the initial condition is defined as $x(0)=u(0)=0$.
Repeating the same construction as previously shown, we obtain that

$$
\begin{equation*}
\left\|A_{\sigma(n+j-1)} A_{\sigma(n+j-2)} \cdots A_{\sigma(j)}\right\| \leq \frac{\left(K_{\sigma}\right)^{k+1}}{\binom{k+n}{k}} \tag{2.10}
\end{equation*}
$$

for any $k \geq 0$.
Introduce the following notation:

$$
\begin{equation*}
\phi_{\sigma}(n)=\sup _{j \geq 0}\left\|A_{\sigma(n+j-1)} A_{\sigma(n+j-2)} \cdots A_{\sigma(j)}\right\| \tag{2.11}
\end{equation*}
$$

for $n \geq 1$.
Lemma 2.2: $\phi_{\sigma}(n)$ tends to zero exponentially fast, say

$$
\begin{equation*}
\phi_{\sigma}(n) \leq c_{\sigma} \lambda_{\sigma}^{n} . \tag{2.12}
\end{equation*}
$$

with some $0<c_{\sigma}<\infty, 0<\lambda_{\sigma}<1$.
Proof: The property $\|A B\| \leq\|A\|\|B\|$ implies that

$$
\begin{equation*}
\phi_{\sigma}(n+m) \leq \phi_{\sigma}(n) \phi_{\sigma}(m) \tag{2.13}
\end{equation*}
$$

and, thus, $\log \phi_{\sigma}(n)$ is a subadditive function. Furthermore, (2.10) implies that $\log \phi_{\sigma}(n)$ tends to $-\infty$. Now it is well known (see [14]) that

$$
\lim \frac{1}{n} \log \phi_{\sigma}(n)=\inf \frac{1}{n} \log \phi_{\sigma}(n)<0
$$

which implies the claim.

It remains to prove that $c_{\sigma}$ and $\lambda_{\sigma}$ can be chosen uniformly with respect to the switching path.
Let us observe that to this aim it is enough to show that

$$
\begin{equation*}
K=\sup _{\sigma} K_{\sigma}<\infty \tag{2.14}
\end{equation*}
$$

where the supremum is taken with respect to all switching paths $\sigma$.
Remark: Note that (2.14) means by definition that $\mathcal{A}$ is productbounded. (see [2]).
Indeed, repeating the arguments in the proof of Lemma 2.1 for any switching path using $K$ instead of $K_{\sigma}$, we get for

$$
\phi(n)=\sup _{\sigma} \sup _{j \geq 0}\left\|A_{\sigma(n+j-1)} A_{\sigma(n+j-2)} \cdots A_{\sigma(j)}\right\|
$$

the upper bound

$$
\begin{equation*}
\phi(n) \leq \frac{K^{k+1}}{\binom{k+n}{k}} \tag{2.15}
\end{equation*}
$$

for any $k \geq 0$. Now, repeating the arguments in the proof of Lemma 2.2 we get $\phi(n)$ converges to zero exponentially fast, and this is the statement of the theorem.
To prove (2.14) an indirect argument will be applied. Assume that there is no universal bound for the operators $T_{\sigma}$. Then, for any sequence $c_{k}, k \geq 1$ tending to infinity we can find a sequence of switching paths $\sigma_{k}$ and input sequences $u_{k}=\left(u_{k}(0), u_{k}(1), \ldots\right)$ with $\sup _{j \geq 0}\left\|u_{k}(j)\right\| \leq 1$ such that for the responses $x_{k}$ defined by

$$
x_{k}=T_{\sigma(k)} u_{k}
$$

we have that $\sup _{n \geq 0}\left\|x_{k}(n)\right\|>c_{k}$. Without violating this property we can also assume that for some finite $n_{k}$ we have $u_{k}(j)=0$ for $j \geq n_{k}$ and

$$
\left\|x_{k}(j)\right\|>c_{k}, \text { for some } 1 \leq j<n_{k}
$$

Also, without loss of generality, we might define $u_{k}(j)=0$, for $j<0$.
Assume now that a sequence $c_{k}$ tending to infinity is given and define a sequence of time indexes $N(k)=n_{1}+\cdots+n_{k}, k \geq 1, N(0)=0$ and the input sequence

$$
u(j)=\sum_{k=0}^{\infty} \lambda_{k+1} u_{k+1}(j-N(k)), \quad j \geq 0
$$

where $\lambda_{k+1}=0$ or 1 to be specified later. Note that for a fixed $j$ only one summand can be different from zero. Obviously, $\sup _{j \geq 0}\|u(j)\| \leq$ 1.

Let us now form the switching path $\sigma$ concatenating $\sigma_{k}, k \geq 1$ taken on the interval $0, \ldots, n_{k}-1$. In other words

$$
\sigma=\left(\sigma_{1}(0), \ldots, \sigma_{1}\left(n_{1}-1\right), \sigma_{2}(0), \ldots, \sigma_{2}\left(n_{2}-1\right), \sigma_{3}(0), \ldots\right) .
$$

We show that we can choose $\lambda_{k}, k \geq 1$ recursively so that

$$
x=T_{\sigma} u
$$

will be an unbounded sequence. Observe that $x(j), j \leq N(k)$ depends only on $\sigma_{j}, u_{j}$ and $\lambda_{j}, j \leq k$.

Assume now that $\lambda_{l}$ has been constructed for $l \leq k$. Set

$$
v_{k}(j)=\sum_{l=0}^{k-1} \lambda_{l+1} u_{l+1}(j-N(l)), \quad j \geq 0
$$

and

$$
z_{k}=T_{\sigma} v_{k}
$$

Then, for $N(k)<j \leq N(k+1)$

$$
x(j)=z_{k}(j)+x_{k+1}(j-N(k)) \lambda_{k+1} .
$$

Now, if

$$
\sup _{N(k)<j \leq N(k+1)}\left\|z_{k}(j)\right\|>\frac{c_{k+1}}{2}
$$

then we set $\lambda_{k+1}=0$. Otherwise, take $\lambda_{k+1}=1$. In both cases

$$
\sup _{N(k)<j \leq N(k+1)}\|x(j)\| \geq \frac{c_{k+1}}{2}
$$

holds. Continuing this way we can construct an unbounded response sequence $x$ generated by a bounded input sequence, contradicting to the BIBO assumption.

Thus, (2.14) holds, concluding the proof of the theorem.
It is worth formulating an immediate consequence of the theorem as a separate corollary.
Corollary 2.1: If the switching system determined by the set of matrices $\mathcal{A}=\left\{A_{\gamma} \mid \gamma \in \Gamma\right\}$ is BIBO stable, then there exists a $\lambda<1$ such that all eigenvalues of the matrices $A_{\gamma}$ are less than or equal to $\lambda$, i.e.,

$$
\left|\lambda_{\max }\left(A_{\gamma}\right)\right| \leq \lambda
$$

where $\lambda_{\max }(A)$ denotes the eigenvalue of the matrix $A$ which has the largest absolute value.

Proof: This is immediate from the observation that choosing the switching path $\sigma$ to be identically $\gamma$ and vanishing the input from the previous theorem, we obtain that
proving the corollary.

## III. Conclusion

We have proved that BIBO stability of a switching linear system implies that it is uniformly exponentially stable. It follows, using a result of Dayawansa and Martin, that the family of transition matrices have a joint, not necessarily quadratic, Lyapunov function. The extension of this result to BIBO stability in a stochastic sense is an open problem, the solution of which would resolve a fundamental technical issue in recursive identification and stochastic adaptive control.

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[^0]:    Manuscript received May 24, 2001; revised April 24, 2002. Recommended by Associate Editor J. M. A. Scherpen. This work was supported in part by the Swedish Research Council for Engineering Sciences (TFR), the Göran Gustafsson Foundation, the National Research Foundation of Hungary (OTKA) under Grants T 015668 and T16665, and the Research Development Grant of the Ministry of Education, Hungary (FKFP) 0194/1997.
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    Publisher Item Identifier 10.1109/TAC.2002.804470.

