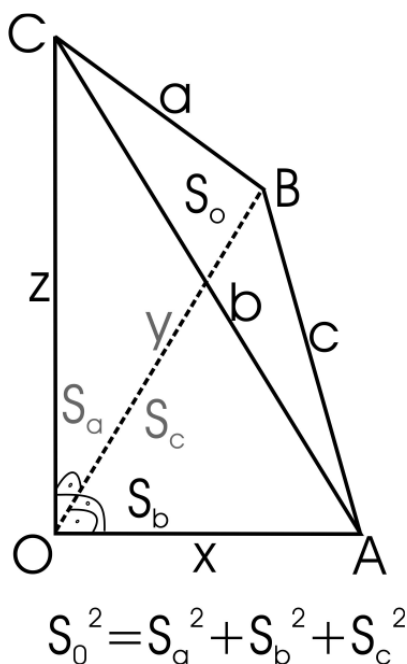


# Pythagorean Theorem in the n-dimensional space

## Pythagorean Theorem in 3 dimension

Let's call  $OABC$  a "rectangular tetrahedron", if both three angles in  $O$  are rectangular.  
 Let's call "legs" the area of the right triangles and "hypotenuse" the area of the  $ABC$  triangle.  
 Then: **The sum of the squares of the "legs" is equal to the square of the "hypotenuse".**

I've discovered this (and generalized it for n-dimensional space) in 1987. Now, dealing with the English translation, looking for English math terms, realized, that this is known as **de Gua's Theorem**:  
 "The square of the area of the base (i.e., the face opposite the right trihedral angle) of a trirectangular tetrahedron is equal to the sum of the squares of the areas of its other three faces.  
 This theorem was presented to the Paris Academy of Sciences in 1783 by J. P. de Gua de Malves."  
 Anyway, I'll continue the translation, to have it all together...



### Notations:

$S_a$  : the area of  $OBC$  triangle  
 $S_b$  : the area of  $OCA$  triangle  
 $S_b$  : the area of  $OAB$  triangle  
 $S_o$  : the area of  $ABC$  triangle  
 $OA = x, OB = y, OC = z, BC = a, CA = b, AB = c$

### Demonstration:

Easy to observe:

$$4S_a^2 = y^2z^2, \quad 4S_b^2 = z^2x^2, \quad 4S_c^2 = x^2y^2, \quad a^2 = y^2 + z^2, \quad b^2 = z^2 + x^2, \quad c^2 = x^2 + y^2$$

Using Lemma 1.:

$$16S_o^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4$$

Substituting:

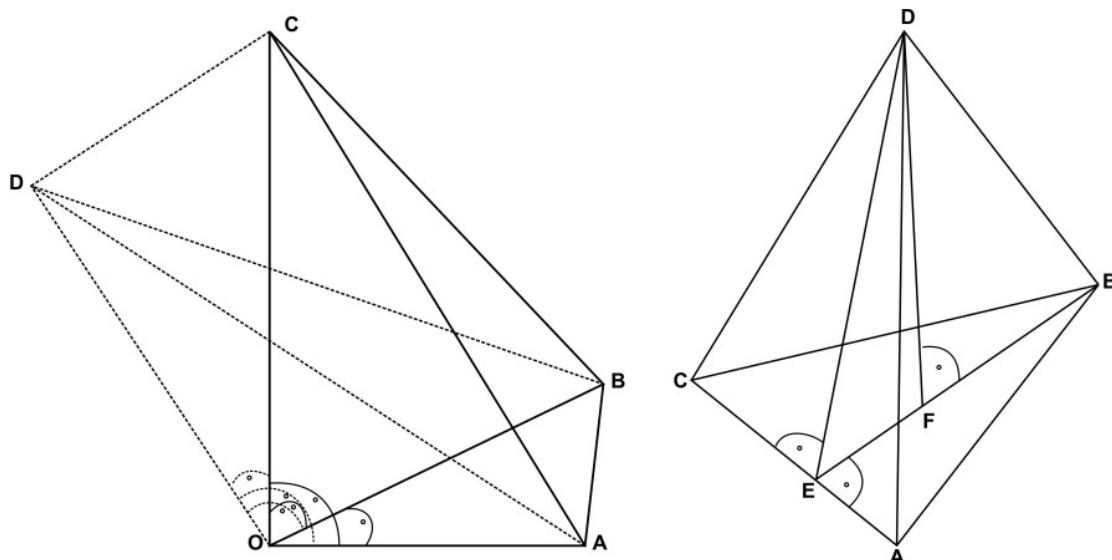
$$\begin{aligned} 16S_o^2 &= 2(y^2 + z^2)(z^2 + x^2) + 2(z^2 + x^2)(x^2 + y^2) + 2(x^2 + y^2)(y^2 + z^2) - (y^2 + z^2)^2 - (z^2 + x^2)^2 - (x^2 + y^2)^2 = \\ &= 2y^2z^2 + 2y^2x^2 + 2z^4 + 2z^2x^2 + 2z^2x^2 + 2z^2y^2 + 2x^4 + 2x^2y^2 + 2x^2y^2 + 2x^2z^2 + 2y^4 + 2y^2z^2 - \\ &- y^4 - 2y^2z^2 - z^4 - z^4 - 2z^2x^2 - x^4 - x^4 - 2x^2y^2 - y^4 = 4y^2z^2 + 4z^2x^2 + 4x^2y^2 = \\ &= 4(4S_a^2) + 4(4S_b^2) + 4(4S_c^2); \end{aligned}$$

So:  $S_o^2 = S_a^2 + S_b^2 + S_c^2$ , q.e.d.

## Pythagorean Theorem in 4 dimension

*The sum of the squares of the “legs” of a four-dimensional “rectangular penta-hedron” is equal to the square of the “hypotenuse”,*

*The “rectangular penta-hedron” is a 4-dimensional object, bounded by four 3-dimensional rectangular tetrahedrons, (their volumes are the “legs”), and by a non-rectangular tetrahedron, (its volume is the “hypotenuse”).*



Each pair of OA, OB, OC, OD segments are perpendiculars.

That is possible only if the vertex D is positioned in an other, in the fourth dimension.

The statement:

$$V^2_{OABC} + V^2_{OABD} + V^2_{OACD} + V^2_{OBCD} = V^2_{ABDC}$$

OA = a, OB = b, OC = c, OD = d =>

$$AB^2 = a^2 + b^2, AC^2 = a^2 + c^2, AD^2 = a^2 + d^2, BC^2 = b^2 + c^2, BD^2 = b^2 + d^2, CD^2 = c^2 + d^2$$

(each of the six OXY triangle is rectangular)

E is the base of the altitude from O to CA. According to Lemma 2., base of the altitude from B to CA is the same point E (OABD is a). In the same way, the base of the altitude from D to CA is also E (OADC is a rectangular tetrahedron). Then:

$$DF \cdot EB = 2S_{EBD}, 2S_{ABC} = AC \cdot EB \Rightarrow S_{ABC} \cdot DF = S_{EBD} \cdot AC$$

$$36V^2_{ABDC} = 4S^2_{ABC} \cdot DF^2 = 4S^2_{EBD} \cdot AC^2 = 4S^2_{EBD} \cdot (a^2 + c^2)$$

We have to calculate  $S^2_{EBD}$

$$BD^2 = b^2 + d^2$$

$$EB^2 = 4S^2_{ABC} / AC^2 = (a^2b^2 + a^2c^2 + b^2c^2) / (a^2 + c^2) \quad (\text{Pythagorean Theorem in 3 dimension})$$

$$ED^2 = 4S^2_{ADC} / AC^2 = (a^2d^2 + a^2c^2 + d^2c^2) / (a^2 + c^2) \quad (\text{Pythagorean Theorem in 3 dimension})$$

Using the modified variant of Lemma 1. for EBD triangle:

$$16S^2_{EBD} \cdot (a^2 + c^2) = 2(a^2d^2 + a^2c^2 + d^2c^2)(b^2 + d^2) + 2(b^2 + d^2)(a^2b^2 + a^2c^2 + b^2c^2) - (b^2 + d^2)^2(a^2 + c^2) -$$

$$- ((a^2d^2 + a^2c^2 + d^2c^2) - (a^2b^2 + a^2c^2 + b^2c^2))^2 / (a^2 + c^2) =$$

$$= 2(a^2d^2 + a^2c^2 + d^2c^2)(b^2 + d^2) + 2(b^2 + d^2)(a^2b^2 + a^2c^2 + b^2c^2) - (b^2 + d^2)^2(a^2 + c^2) -$$

$$- ((a^2 + c^2)(d^2 - b^2))^2 / (a^2 + c^2) =$$

$$= 2a^2d^2b^2 + 2a^2c^2b^2 + 2d^2c^2b^2 + 2a^2d^2d^2 + 2a^2c^2d^2 + 2d^2c^2d^2 + 2b^2a^2b^2 + 2b^2a^2c^2 + 2b^2b^2c^2 + 2d^2a^2b^2 +$$

$$+ 2d^2a^2c^2 + 2d^2b^2c^2 - (b^2 + d^2)^2(a^2 + c^2) - (a^2 + c^2)(b^2 - d^2)^2 =$$

$$= 4a^2b^2d^2 + 4a^2b^2c^2 + 4a^2c^2d^2 + 4b^2c^2d^2 + 2a^2d^4 + 2c^2d^4 + 2b^4a^2 + 2b^4c^2 - (2b^4 + 2d^4)(a^2 + c^2) =$$

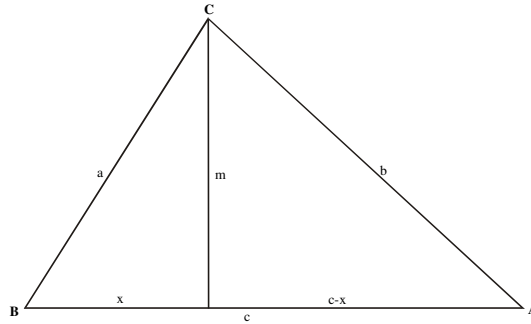
$$= 4 \cdot 36 \cdot (V^2_{OABD} + V^2_{OABC} + V^2_{OACD} + V^2_{OBCD})$$

$$\text{thus } 4S^2_{EBD} \cdot (a^2 + c^2) = 36(V^2_{OABC} + V^2_{OABD} + V^2_{OACD} + V^2_{OBCD})$$

$$\text{that is } V^2_{OABC} + V^2_{OABD} + V^2_{OACD} + V^2_{OBCD} = V^2_{ABDC} \text{ q.e.d.}$$

## Lemma 1.

Using the figure's notations (S denotes the area of the rectangular triangle):



$$\underline{16S^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}$$

### Demonstration:

$$a^2 - x^2 = b^2 - (c-x)^2 = m^2 \Rightarrow 2cx = a^2 - b^2 + c^2$$

then again

$$m^2 = a^2 - x^2 \text{ and } 4S^2 = m^2c^2,$$

and substituting:

$$16S^2 = 4m^2c^2 = 4a^2c^2 - 4c^2x^2 = 4a^2c^2 - (a^2 - b^2 + c^2)^2 = 4a^2c^2 - a^4 - b^4 - c^4 + 2a^2b^2 + 2b^2c^2 - 2c^2a^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4, \text{ q.e.d.}$$

We will use a slightly modified variant too:

$$16S^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4 = 2a^2b^2 + 2b^2c^2 - b^4 - (a^4 - 2c^2a^2 + c^4) = 2a^2b^2 + 2b^2c^2 - b^4 - (a^2 - c^2)^2$$

## Lemma 2.

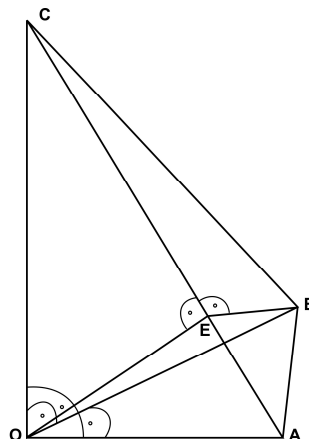
The base (E) of the altitude from the rectangular vertex (O) of a rectangular tetrahedron (OABC) to an opposite edge (e.g. AC) is identical to the base of the altitude from the other vertex opposite to that edge (i.e. B).

### Justification:

The OB segment is perpendicular to the OAC plane, so is perpendicular to the AC segment (because it is contained in that plane).

The OE segment is perpendicular to the AC segment (because the tetrahedron is rectangular).

Therefore AC is perpendicular to the BOE plane, so is perpendicular to the BE segment (because it is contained in that plane), so E will be the base of the altitude from B to AC.



# Pythagorean Theorem in the n-dimensional space

An n-dimensional rectangular “(n+1)-hedron” is defined by the O, A<sub>1</sub>, A<sub>2</sub>,...A<sub>n</sub> points, so that for each pair OA<sub>i</sub> is perpendicular to OA<sub>j</sub>. That is bounded by n rectangular (n-1)-dimensional “n-hedrons” (OA<sub>1</sub>A<sub>2</sub>...A<sub>n-1</sub>, OA<sub>2</sub>A<sub>3</sub>...A<sub>n</sub>,...), and an (n-1)-dimensional non-rectangular “n-hedron” (A<sub>1</sub>A<sub>2</sub>...A<sub>n</sub>).

Let's call the (n-1)-dimensional volume of rectangular “n-hedrons” “legs”, and the volume of non-rectangular one “hypotenuse”. Then the theorem says:

***The sum of the squares of the “legs” is equal to the square of the “hypotenuse”.***

## The volume of multi-dimensional solids

We can accept, that the volume of an n-dimensional “cube” is a<sup>n</sup>. (Let's chose the one defined by the same O, A<sub>1</sub>, A<sub>2</sub>,...A<sub>n</sub> points.) A such a “cube” can be split into n congruent n-dimensional rectangular “pyramids”. The basis of them are the (n-1)-dimensional “cubes” (e.g. those containing O) and their vertex is the point opposite to O. (e.g. if the coordinates of O is (0,0,...,0), the opposite point is (1,1,...,1)). The volume of such a “pyramid” then is the volume of the cube divided by n. So we can state, that *the volume of such a “pyramid” is the volume of its (n-1)-dimensional “base” multiplied by its “height” an divided by n.* The concept of the “height” for orthogonal “pyramids” (spited from orthogonal “prism” is trivial, for other “pyramids” I'll specify.

From this we can easily deduce, that the volume of an n-dimensional rectangular “(n+1)-hedron” is a<sub>1</sub>\*a<sub>2</sub>\*...\*a<sub>n</sub>/n!, where a<sub>1</sub>,a<sub>2</sub>,...,a<sub>n</sub>, are the length of the edges starting from the rectangular point.

## Demonstration

We will demonstrate the theorem with mathematical induction. Assume, that the theorem is true for the cases from 2 to (n-1), and based on that we will demonstrate for next number: n.

The n-dimensional rectangular “(n+1)-hedron” is defined by the points O, A<sub>1</sub>, A<sub>2</sub>,... A<sub>n</sub>, their coordinates are: (0,0,...,0), (a<sub>1</sub>,0,...,0), (0,a<sub>2</sub>,...,0),..., (0,0,...,0,a<sub>n</sub>). The “leg” i is defined by all the points but A<sub>i</sub> (i=1, 2, ..., n), the “hypotenuse” is defined by all the point but O.

So the “leg” i is Pa/a<sub>i</sub>/(n-1)!, where Pa=a<sub>1</sub>\*a<sub>2</sub>\*...\*a<sub>n</sub>

We will consider the (n-1)-dimensional non-rectangular “n-hedron” (A<sub>1</sub>A<sub>2</sub>...A<sub>n</sub>) as a (n-1)-dimensional “pyramid”. Its base is the (n-2)-dimensional (n-1)-hedron defined by the A<sub>1</sub>, A<sub>2</sub>,...A<sub>n-1</sub> points, and its vertex is A<sub>n</sub>. The height (h) of it is the hypotenuse of a triangle. One leg is OA<sub>n</sub>, the other one is the height (h<sub>0</sub>) of (n-1)-dimensional rectangular “n-hedron” (OA<sub>1</sub>A<sub>2</sub>...A<sub>n-1</sub>). Using the theorem for (n-1), the square of its “hypotenuse”, (which is the “base” of the A<sub>1</sub>A<sub>2</sub>...A<sub>n</sub> “pyramid”) is:

$$V_{n-2}^2 = Pa^2/a_n^2 * (1/a_1^2 + 1/a_2^2 + ... + 1/a_{n-1}^2) / ((n-2)!)^2$$

The volume of the (n-1)-dimensional rectangular “n-hedron” (OA<sub>1</sub>A<sub>2</sub>...A<sub>n-1</sub>).

$$V_{n-2}^2 * m_0^2 / (n-1)^2 = V_{n-1}^2 \text{ so the “height” of } OA_1A_2...A_{n-1} \text{ (the distance of O from the } A_1A_2...A_{n-1} \text{ points):}$$

$$h_0^2 = V_{n-1}^2 * (n-1)^2 / V_{n-2}^2 = Pa^2/a_n^2 / ((n-1)!)^2 * (n-1)^2 / (Pa^2/a_n^2 * (1/a_1^2 + 1/a_2^2 + ... + 1/a_{n-1}^2) / ((n-2)!)^2)$$

$$h_0^2 = 1 / (1/a_1^2 + 1/a_2^2 + ... + 1/a_{n-1}^2) \text{ (interesting...)}$$

So the square of the distance of A<sub>n</sub> from the A<sub>1</sub>A<sub>2</sub>...A<sub>n-1</sub> points:

$$h^2 = h_0^2 + a_n^2, \text{ so the square of the volume of } A_1A_2...A_n:$$

$$V^2(A_1A_2...A_n) = h^2 * V_{n-2}^2 / (n-1)^2 =$$

$$= (1 / (1/a_1^2 + 1/a_2^2 + ... + 1/a_{n-1}^2) + a_n^2) * Pa^2/a_n^2 * (1/a_1^2 + 1/a_2^2 + ... + 1/a_{n-1}^2) / ((n-2)!)^2 / (n-1)^2 =$$

$$= (Pa^2/a_n^2 + Pa^2/a_1^2 + Pa^2/a_2^2 + ... + Pa^2/a_{n-1}^2) / ((n-1)!)^2$$

what is the sum of the square of the “legs” of the n-dimensional (n-1)-hedron. , q.e.d.

## Note

Consider **n<sub>1</sub>, n<sub>2</sub>, ..., n<sub>n</sub>** the normal vectors of the “legs” and **n<sub>0</sub>** the normal vector of “hypotenuse”. Assume that they are proportional with the respective “volumes”, or in this case the “surfaces” is the better word. The sum of the vectors must be 0, because those are the components of a closed surface:

$$\mathbf{n}_1 + \mathbf{n}_2 + \dots + \mathbf{n}_n = -\mathbf{n}_0$$

On the other hand, each pair of the **n<sub>1</sub>, n<sub>2</sub>, ..., n<sub>n</sub>** are perpendiculars, so:

$$|\mathbf{n}_1 + \mathbf{n}_2 + \dots + \mathbf{n}_n|^2 = n_1^2 + n_2^2 + \dots + n_n^2 = n_0^2$$

That's an easier way to demonstrate.

*The idea arose at autumn 1987(?), the translation was finished at 24 January 2010 in Budapest.*

*Istvan Meder*